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ON THE CODEGREE DENSITY OF $PG_m(q)$*

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Abstract. For an r-graph G, the minimum $(r-1)$-degree $\delta(G)$ is the largest integer t such that every $(r-1)$-subset of $V(G)$ is contained in at least t edges of G. Given an r-graph F, the codegree density $\gamma(F)$ is the largest $\gamma > 0$ such that there are F-free r-graphs G on n vertices with $\delta(G) \geq (\gamma - o(1))n$. In this paper, we consider the codegree density of projective geometries. Employing the moment identity of a subset of $PG_m(q)$, we prove (1) $\gamma(PG_2(q)) = \frac{1}{2}$ for prime power $q \equiv 2 \pmod 3$; and (2) $\gamma(PG_3(q)) = \frac{2}{3}$ for prime power $q \equiv 1 \pmod 2$ or $q \equiv 2 \pmod 3$. Our results partially solve an open problem proposed by Keevash and Zhao [*J. Combin. Theory Ser. B*, 97 (2007), pp. 919–928]. Previously, the codegree density problems for projective geometries were settled only for $PG_2(2)$, $PG_3(2)$, $PG_3(3)$, and $PG_2(q)$ with odd prime power q.

Key words. codegree, hypergraph, projective geometry

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1. Introduction. In this paper, an r-graph is always an r-uniform hypergraph. Let H be an r-graph. An r-graph G is called H-free if G contains no copy of H as its subhypergraph. The *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in an n-vertex H-free r-graph. The determination of the Turán number $\mathrm{ex}(n, H)$ is a fundamental problem in extremal combinatorics. An easy averaging argument shows that the nonnegative sequence $\mathrm{ex}(n, H)/\binom{n}{r}$ is nonincreasing and hence converges to a limit as n goes to infinity. This limit is usually called *Turán density* of H and is denoted by $\pi(H)$.

For simple graphs ($r = 2$), Turán [16] determined the exact value of $\mathrm{ex}(n, K_t)$ for all complete graphs K_t. Later, Erdős and Stone [2] showed that $\pi(H) = 1 - \frac{1}{\chi(H)-1}$, where $\chi(H)$ denotes the chromatic number of H. In contrast to the simple graphs, there are only a few results for the hypergraph Turán problems. For example, even the value of $\pi(K_t^{(r)})$ is still unknown for any $t > r > 2$, where $K_t^{(r)}$ denotes the complete r-graph on t vertices. It is conjectured that $\pi(K_4^{(3)}) = \frac{5}{9}$. Recently, there has been some new progress for the hypergraph Turán problems (for example, see [5, 6, 7, 10, 14, 17]). For more extremal results of hypergraphs, we refer the reader to the survey [9].

A natural variation of Turán problem is to consider how large the minimum degree can be in an H-free r-graph. For any r-graph G, let $\delta(G)$ be the minimum $d(S)$ of a set S of $r - 1$ vertices, where $d(S)$ is the number of edges of G containing S. The codegree extremal number $\text{co-ex}(n, H)$ is the maximum possible value of $\delta(G)$, where

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G is an H-free r-graph on n vertices. In [15], Mubayi and Zhao showed that

$$\gamma(H) = \lim_{n\to\infty} \frac{\text{co-ex}(n, H)}{n}$$

exists, which is called the *codegree density* of H.

It seems that for most hypergraphs, the codegree Turán problems are not easier than the original Turán problems. There are only a few r-graphs for which the codegree densities have been determined. In [13], Mubayi showed that $\gamma(PG_2(2)) = \frac{1}{2}$. Later, DeBiasio and Jiang [1] gave an alternative proof for codegree threshold for the Fano plane. Keevash and Zhao [11] studied the codegree density for other projective geometries and constructed a family of 3-graphs whose codegree densities are $1 - \frac{1}{t}$ for all integers $t > 1$. In [3], Falgas-Ravry et al. determined $\gamma(F_{3,2})$, where $F_{3,2}$ is the 3-graph on $\{1, 2, 3, 4, 5\}$ with edges $123, 124, 125, 345$. Falgas-Ravry et al. [4] proved that $\gamma(K_4^{3-}) = \frac{1}{4}$, where K_4^{3-} is the 3-graphs on four vertices with three edges. In [12], Lo and Zhao considered the codegree density of complete r-graphs.

In this paper, we focus on the codegree problem for projective geometries. The following result can be found in [11].

Theorem 1.1 (see [11]). *The codegree density of projective geometries satisfies $\gamma(PG_m(q)) \le 1 - \frac{1}{m}$. Equality holds whenever $m = 2$ and q is 2 or odd, and whenever $m = 3$ and q is 2 or 3.*

The behavior of $\gamma(PG_2(4))$ seems different from others, and Keevash and Zhao [11] proved that

$$\gamma(PG_2(4)) \ge \frac{1}{3}.$$

Later, by the hypergraph regularity method, Keevash [8] showed the following upper bound:

$$\gamma(PG_2(4)) < \frac{1}{2} - \epsilon$$

for some $\epsilon > 0$.

Keevash and Zhao [11] also proposed the following open problem.

Problem 1.2. *Is $\gamma(PG_m(q)) > 0$ for all m and q?*

In this paper, we will continue this investigation and determine more values of $\gamma(PG_m(q))$. For $m = 2$, we prove the following result.

Theorem 1.3. *Let q be a prime power with $q \equiv 2 \pmod 3$; then we have*

$$\gamma(PG_2(q)) = \frac{1}{2}.$$

In particular, we have $\gamma(PG_2(2^{2k+1})) = \frac{1}{2}$ for any integer $k \ge 0$. For $m = 3$, we have the following result.

Theorem 1.4. *Let q be a prime power with $q \equiv 1 \pmod 2$ or $q \equiv 2 \pmod 3$; then we have*

$$\gamma(PG_3(q)) = \frac{2}{3}.$$

The rest of this paper is organized as follows. In section 2, we give some basics of projective geometries. In section 3, we prove Theorem 1.3. Section 4 contains the proof of Theorem 1.4. Section 5 concludes this paper.

2. Preliminaries. Let $\mathbb{F}_q$ denote the finite field with q elements. The projective geometry of dimension m over $\mathbb{F}_q$, denoted by $PG_m(q)$, is the following $(q+1)$-graph. Its vertex set is the set of all one-dimensional subspaces of $\mathbb{F}_q^{m+1}$. Its edges are the two-dimensional subspaces of $\mathbb{F}_q^{m+1}$, in which for each two-dimensional subspace, the set of one-dimensional subspaces that it contains is an edge of the hypergraph $PG_m(q)$.

Let K be a subset of $PG_m(q)$; a line ℓ of $PG_m(q)$ is an i-secant of K if $|\ell \cap K| = i$. Let τ_i denote the total number of i-secants to K; then we have the following lemma.

LEMMA 2.1. *Let K be a subset of $PG_m(q)$ and let $|K| = k$; then the following equations hold:*

$$\sum_{i=0}^{q+1} \tau_i = \frac{(q^m-1)(q^{m+1}-1)}{(q-1)(q^2-1)},$$
$$\sum_{i=1}^{q+1} i\tau_i = k\sum_{i=0}^{m-1} q^i,$$
$$\sum_{i=2}^{q+1} \binom{i}{2}\tau_i = \binom{k}{2}.$$

Proof. The equations express in different ways the cardinality of the following sets:
1. $\{\ell : \ell \text{ is a line of } PG_m(q)\}$;
2. $\{(P,\ell) : P \in (K\cap\ell), \ell \text{ is a line of } PG_m(q)\}$;
3. $\{(\{P,Q\},\ell) : P,Q \in (K\cap\ell), \ell \text{ is a line of } PG_m(q)\}$. ◻

3. Dimension 2: Proof of Theorem 1.3. Let $q \equiv 2 \pmod 3$. By Theorem 1.1, it suffices to show that $\gamma(PG_2(q)) \geq 1/2$. Let V be a set of n vertices. Partition it as $V = V_1 \cup V_2$ so that $||V_i| - \frac{n}{2}| < 1$ for $i = 1,2$. Let G be the $(q+1)$-graph such that if e is an edge of G, then $(|e\cap V_1|, |e\cap V_2|) \in \{(i, q+1-i) : i \not\equiv 0 \pmod 3, 1 \leq i \leq q\}$. Then it is easy to see that $\delta(G) > (\frac{1}{2} - o(1))n$.

Suppose G contains a copy of $PG_2(q)$. Let $K = PG_2(q) \cap V_1$ and let $k = |K|$. For $1 \leq i \leq q$, let τ_i denote the total number of i-secants to K. Then by Lemma 2.1, we have

$$\sum_{i\not\equiv 0 \pmod 3} \tau_i = q^2+q+1, \tag{1}$$
$$\sum_{i\not\equiv 0 \pmod 3} i\tau_i = k(q+1), \tag{2}$$
$$\sum_{i\not\equiv 0 \pmod 3} \binom{i}{2}\tau_i = \binom{k}{2}. \tag{3}$$

Note that

$$\binom{s}{2} \pmod 3 \equiv \begin{cases} 0 & \text{if } s \equiv 0 \text{ or } 1 \pmod 3, \\ 1 & \text{if } s \equiv 2 \pmod 3. \end{cases}$$

Taking both sides of (1), (2), and (3) modulo 3, we obtain

$$\sum_{i\equiv 1 \pmod 3} \tau_i + \sum_{i\equiv 2 \pmod 3} \tau_i \equiv 1 \pmod 3, \tag{4}$$

$$\sum_{i\equiv 1 \pmod 3} \tau_i - \sum_{i\equiv 2 \pmod 3} \tau_i \equiv 0 \pmod 3, \tag{5}$$

$$\sum_{i\equiv 2 \pmod 3} \tau_i \equiv \binom{k}{2} \pmod 3. \tag{6}$$

From (4) and (5), we obtain

$$2 \sum_{i\equiv 2 \pmod 3} \tau_i \equiv 1 \pmod 3.$$

Combining with (6), we have $\binom{k}{2} \equiv 2 \pmod 3$, which is a contradiction.

4. Dimension 3: Proof of Theorem 1.4. In this section, we will determine the values of $\gamma(PG_3(q))$ for prime power q with $q \equiv 2 \pmod 3$ or $q \equiv 1 \pmod 2$. By Theorem 1.1, it suffices to show that $\gamma(PG_3(q)) \geq 2/3$. We divide our discussion into three cases.

4.1. $q \equiv 2 \pmod 3$. Let V be a set of n vertices. Partition it as $V = V_1 \cup V_2 \cup V_3$ so that $||V_i| - \frac{n}{3}| < 1$ for $i = 1, 2, 3$. We say that a set A of vertices has type (a, b, c) if $(|A \cap V_1| \pmod 3, |A \cap V_2| \pmod 3, |A \cap V_3| \pmod 3) = (a, b, c)$. We define a $(q+1)$-graph G on V as follows. The edges are all the sets of $q+1$ vertices of V having type in S, where

$$S = \{(0, 1, 2), (0, 2, 1), (1, 2, 0), (1, 0, 2), (2, 1, 0), (2, 0, 1)\}.$$

First, we verify the codegree property, i.e., all possible types (a, b, c) for a set of q vertices can be obtained from a type of G by decreasing one number by 1. Noting that $q \equiv 2 \pmod 3$, it is easy to see that (a, b, c) is a permutation of $(1, 1, 0), (2, 0, 0)$, or $(1, 2, 2)$. Since S is closed under permutation, we will only consider one case of permutations of (a, b, c). If $(a, b, c) = (1, 1, 0)$, then it can be obtained from $(2, 1, 0)$ or $(1, 2, 0)$. If $(a, b, c) = (2, 0, 0)$, then it can be obtained from $(2, 1, 0)$ or $(2, 0, 1)$. If $(a, b, c) = (1, 2, 2)$, then it can be obtained from $(1, 0, 2)$ or $(1, 2, 0)$. Hence $\delta(G) > (\frac{2}{3} - o(1))n$.

Suppose G contains a copy of $PG_3(q)$. Let $K_i = PG_3(q) \cap V_i$ and let $k_i = |K_i|$. For $0 \leq j \leq q$, let τ_{ij} denote the total number of j-secants to K_i. Then by Lemma 2.1, we have

$$\sum_{j=2}^{q} \binom{j}{2} \tau_{ij} = \binom{k_i}{2}.$$

Let

$$L_i = \{e : e \text{ is a line in } PG_3(q),\ |e \cap K_i| \equiv 2 \pmod 3\}.$$

Then it is easy to see that L_1, L_2, L_3 form a partition of the lines of $PG_3(q)$. Hence, we have

$$|L_1| + |L_2| + |L_3| = (q^2 + 1)(q^2 + q + 1) \equiv 2 \pmod 3.$$

On the other hand, $|L_i| = \sum_{j\equiv 2 \pmod 3} \tau_{ij} \equiv \sum_{j=2}^{q} \binom{j}{2}\tau_{ij} \equiv \binom{k_i}{2} \pmod 3$. Then we have

$$2 \equiv |L_1| + |L_2| + |L_3| \equiv \binom{k_1}{2} + \binom{k_2}{2} + \binom{k_3}{2} \pmod 3. \tag{7}$$

Noting that K_1, K_2, K_3 partition the vertices of $PG_3(q)$, then we have

$$k_1 + k_2 + k_3 = q^3 + q^2 + q + 1 \equiv 0 \pmod 3. \tag{8}$$

Equation (8) implies $k_1 \equiv k_2 \equiv k_3 \pmod 3$ or $\{k_1, k_2, k_3\} \pmod 3 = \{0, 1, 2\}$, which contradicts (7).

4.2. $q \equiv 3 \pmod 4$. Let V be a set of n vertices. Partition it as $V = V_1 \cup V_2 \cup V_3$ so that $||V_i| - \frac{n}{3}| < 1$ for $i = 1, 2, 3$. We say that a set A of vertices has type (a, b, c) if $(|A \cap V_1| \pmod 4, |A \cap V_2| \pmod 4, |A \cap V_3| \pmod 4) = (a, b, c)$. We define a $(q+1)$-graph G on V as follows. The edges are all the sets of $q+1$ vertices of V having type in S, where

$$\begin{aligned} S =&\{(3,1,0),(3,0,1),(1,3,0),(1,0,3),(0,1,3),(0,3,1),(2,1,1),(1,2,1),(1,1,2),\\ &(3,3,2),(3,2,3),(2,3,3)\}. \end{aligned}$$

First, we verify the codegree property, i.e., all possible types (a, b, c) for a set of q vertices can be obtained from a type of G by decreasing one number by 1. Noting that $q \equiv 3 \pmod 4$, it is easy to see that (a, b, c) is a permutation of $(1, 1, 1), (2, 1, 0), (3, 0, 0)$, $(3, 2, 2)$, or $(3, 3, 1)$. Since S is closed under permutation, we will only consider one case of permutations of (a, b, c). If $(a, b, c) = (1, 1, 1)$, it can be obtained from $(2, 1, 1), (1, 2, 1)$, or $(1, 1, 2)$. If $(a, b, c) = (2, 1, 0)$, then it can be obtained from $(3, 1, 0)$ or $(2, 1, 1)$. If $(a, b, c) = (3, 0, 0)$, then it can be obtained from $(3, 1, 0)$ or $(3, 0, 1)$. If $(a, b, c) = (3, 2, 2)$, then it can be obtained from $(3, 3, 2)$ or $(3, 2, 3)$. If $(a, b, c) = (3, 3, 1)$, then it can be obtained from $(3, 0, 1)$, $(0, 3, 1)$, or $(3, 3, 2)$. Hence $\delta(G) > (\frac{2}{3} - o(1))n$.

Suppose G contains a copy of $PG_3(q)$. Let H be the set of vertices of any hyperplane of $PG_3(q)$ and $H_i = H \cap V_i$ for $i = 1, 2, 3$. We claim that each $|H_i|$ is odd. Since H_1, H_2, H_3 partition the vertices of H, then we have

$$|H_1| + |H_2| + |H_3| = q^2 + q + 1 \equiv 1 \pmod 4. \tag{9}$$

Let

$$\begin{aligned} L_i &= \{e : e \text{ is a line in } H,\ |e \cap H_i| \pmod 4 \in \{2, 3\}\},\\ L_{233} &= \{e : e \text{ is a line in } H, \text{ and has type } (2,3,3), (3,2,3), \text{ or } (3,3,2)\}. \end{aligned}$$

Then $L_{233} \in L_i$ for $i = 1, 2, 3$ and the sets $L_1 \backslash L_{233}$, $L_2 \backslash L_{233}$, $L_3 \backslash L_{233}$, and L_{233} form a partition of the lines of H. Hence we have

$$\begin{aligned} q^2 + q + 1 &= |L_1 \backslash L_{233}| + |L_2 \backslash L_{233}| + |L_3 \backslash L_{233}| + |L_{233}|\\ &= |L_1| + |L_2| + |L_3| - 2|L_{233}|. \end{aligned}$$

Therefore,

$$|L_1| + |L_2| + |L_3| \equiv 1 \pmod 2.$$

Let $\ell_j(H_i)$ denote the number of lines with exactly j points in H_i. Then $|L_i| = \sum_{j\equiv 2,3 \pmod 4} \ell_j(H_i)$. On the other hand, by Lemma 2.1,

$$\binom{|H_i|}{2} = \sum_{j\geq 2} \binom{j}{2} \ell_j(H_i). \tag{10}$$

Note that

$$\binom{s}{2} \pmod 2 \equiv \begin{cases} 0 & \text{if } s \equiv 0 \text{ or } 1 \pmod 4, \\ 1 & \text{if } s \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Taking both sides of (10) modulo 2, we obtain

$$\begin{aligned} \binom{|H_i|}{2} &\equiv \sum_{j\equiv 2,3 \pmod 4} \ell_j(H_i) \pmod 2 \\ &\equiv |L_i| \pmod 2. \end{aligned}$$

Hence, we obtain the following equations:

$$\begin{aligned} &|H_1| + |H_2| + |H_3| = q^2 + q + 1 \equiv 1 \pmod 4, \\ &\binom{|H_1|}{2} + \binom{|H_2|}{2} + \binom{|H_3|}{2} \equiv |L_1| + |L_2| + |L_3| \equiv 1 \pmod 2. \end{aligned}$$

Considering Table 1, where $|H_1| + |H_2| + |H_3| \equiv 1 \pmod 4$, then we have

$$|H_1| \equiv |H_2| \equiv |H_3| \equiv 1 \pmod 2. \tag{11}$$

TABLE 1

$\{\lvert H_1\rvert, \lvert H_2\rvert, \lvert H_3\rvert\} \pmod 4$	$\binom{\lvert H_1\rvert}{2} + \binom{\lvert H_2\rvert}{2} + \binom{\lvert H_3\rvert}{2} \equiv 1 \pmod 2$
$\{0, 0, 1\}$	0
$\{0, 2, 3\}$	0
$\{1, 1, 3\}$	1
$\{1, 2, 2\}$	0

Let $U_i = V(PG_3(q)) \cap V_i$ for $i = 1, 2, 3$, where $V(PG_3(q))$ is the vertex set of graph $PG_3(q)$. Fixing a point $x \in U_1$, we count the size of the following set:

$$\{(y, H) : \ y \in U_2, \ x, y \in H, \ H \text{ is a hyperplane of } PG_3(q)\}.$$

For each y, there are $q + 1$ hyperplanes containing x and y, so the number of pairs (y, H) is $(q + 1)|U_2|$, which is even. On the other hand, there are $q^2 + q + 1$ planes containing x, and each contains an odd number of points from U_2 by the above claim (see (11)), which implies that the number of pairs (y, H) is odd, a contradiction.

4.3. $q \equiv 1 \pmod 4$. Let V be a set of n vertices. Partition it as $V = V_1 \cup V_2 \cup V_3$ so that $||V_i| - \frac{n}{3}| < 1$ for $i = 1, 2, 3$. We say that a set A of vertices has type (a, b, c) if $(|A \cap V_1| \pmod 4, |A \cap V_2| \pmod 4, |A \cap V_3| \pmod 4) = (a, b, c)$. We define a $(q+1)$-graph G on V as follows. The edges are all the sets of $q + 1$ vertices of V having type in S, where

$$\begin{aligned} S = &\{(0, 1, 1), (1, 0, 1), (1, 1, 0), (0, 3, 3), (3, 0, 3), (3, 3, 0), (1, 2, 3), (1, 3, 2), (2, 1, 3), \\ &(2, 3, 1), (3, 1, 2), (3, 2, 1)\}. \end{aligned}$$

First, we verify the codegree property, i.e., all possible types (a,b,c) for a set of q vertices can be obtained from a type of G by decreasing one number by 1. Noting that $q \equiv 1 \pmod 4$, it is easy to see that (a,b,c) is a permutation of $(1,0,0),(1,1,3),(1,2,2)$, $(2,0,3)$, or $(3,3,3)$. Since S is closed under permutation, we will only consider one case of permutations of (a,b,c). If $(a,b,c)=(1,0,0)$, it can be obtained from $(1,0,1)$ or $(1,1,0)$. If $(a,b,c)=(1,1,3)$, then it can be obtained from $(1,1,0),(1,2,3)$, or $(2,1,3)$. If $(a,b,c)=(1,2,2)$, then it can be obtained from $(1,2,3)$ or $(1,3,2)$. If $(a,b,c)=(2,0,3)$, then it can be obtained from $(2,1,3)$ or $(3,0,3)$. If $(a,b,c)=(3,3,3)$, then it can be obtained from $(0,3,3),(3,0,3)$, or $(3,3,0)$. Hence $\delta(G) > (\frac{2}{3}-o(1))n$.

Suppose G contains a copy of $PG_3(q)$. Let H be the set of vertices of any hyperplane of $PG_3(q)$ and $H_i = H \cap V_i$ for $i=1,2,3$. We claim that each $|H_i|$ is odd. Since H_1,H_2,H_3 partition the vertices of H, we then have

$$|H_1|+|H_2|+|H_3| = q^2+q+1 \equiv 3 \pmod 4. \tag{12}$$

Let

$$\begin{aligned} L_i &= \{e : e \text{ is a line in } H,\ |e\cap H_i| \pmod 4 \in \{0,1\}\},\\ L_{011} &= \{e : e \text{ is a line in } H, \text{ and has type } (0,1,1),(1,0,1), \text{ or } (1,1,0)\}. \end{aligned}$$

Then $L_{011} \in L_i$ for $i=1,2,3$ and the sets $L_1\backslash L_{011}$, $L_2\backslash L_{011}$, $L_3\backslash L_{011}$, and L_{011} form a partition of the lines of H. Therefore, we have

$$\begin{aligned} q^2+q+1 &= |L_1\backslash L_{011}|+|L_2\backslash L_{011}|+|L_3\backslash L_{011}|+|L_{011}|\\ &= |L_1|+|L_2|+|L_3|-2|L_{011}|. \end{aligned}$$

Hence

$$|L_1|+|L_2|+|L_3| \equiv 1 \pmod 2.$$

Let $\ell_j(H_i)$ denote the number of lines with exactly j points in H_i. Then $|L_i| = \sum_{j\equiv 0,1 \pmod 4} \ell_j(H_i)$. On the other hand, by Lemma 2.1, we have

$$\sum_{j=0}^{q+1} \ell_j(H_i) = q^2+q+1, \tag{13}$$

$$\sum_{j\ge 2} \binom{j}{2} \ell_j(H_i) = \binom{|H_i|}{2}. \tag{14}$$

Note that

$$\binom{s}{2} \pmod 2 \equiv \begin{cases} 0 & \text{if } s\equiv 0 \text{ or } 1 \pmod 4,\\ 1 & \text{if } s \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

Taking both sides of (14) modulo 2, we obtain

$$\binom{|H_i|}{2} \equiv \sum_{j\equiv 2,3 \pmod 4} \ell_j(H_i) \pmod 2.$$

Then we have

$$\begin{aligned}|L_i| &= \sum_{j\equiv 0,1 \pmod 4} \ell_j(H_i)\\ &= \sum_{j=0}^{q+1} \ell_j(H_i) - \sum_{j\equiv 2,3 \pmod 4} \ell_j(H_i)\\ &\overset{(13)}{=} q^2+q+1 - \sum_{j\equiv 2,3 \pmod 4} \ell_j(H_i)\\ &\equiv 1 - \binom{|H_i|}{2} \pmod 2.\end{aligned}$$

Hence, we obtain the following equations:

$$|H_1|+|H_2|+|H_3| = q^2+q+1 \equiv 3 \pmod 4,$$
$$\binom{|H_1|}{2}+\binom{|H_2|}{2}+\binom{|H_3|}{2} \equiv 3-(|L_1|+|L_2|+|L_3|) \equiv 0 \pmod 2.$$

Considering Table 2, where $|H_1|+|H_2|+|H_3| \equiv 3 \pmod 4$, we then have

(15) $$|H_1| \equiv |H_2| \equiv |H_3| \equiv 1 \pmod 2.$$

Table 2

$\{\lvert H_1\rvert, \lvert H_2\rvert, \lvert H_3\rvert\} \pmod 4$	$\binom{\lvert H_1\rvert}{2}+\binom{\lvert H_2\rvert}{2}+\binom{\lvert H_3\rvert}{2} \equiv 1 \pmod 2$
$\{0,0,3\}$	1
$\{0,1,2\}$	1
$\{1,1,1\}$	0
$\{1,3,3\}$	0
$\{2,2,3\}$	1

Let $U_i = V(PG_3(q)) \cap V_i$ for $i = 1,2,3$, where $V(PG_3(q))$ is the vertex set of graph $PG_3(q)$. Fixing a point $x \in U_1$, we count the size of the following set:

$$\{(y,H):\ y \in U_2,\ x,y \in H,\ H \text{ is a hyperplane of } PG_3(q)\}.$$

For each y, there are $q+1$ hyperplanes containing x and y, so the number of pairs (y,H) is $(q+1)|U_2|$, which is even. On the other hand, there are q^2+q+1 planes containing x, and each contains an odd number of points from U_2 by the above claim (see (15)), which implies that the number of pairs (y,H) is odd, a contradiction.

5. Conclusion. If $a \le b$, then $PG_a(q) \subseteq PG_b(q)$, hence $\gamma(PG_a(q)) \le \gamma(PG_b(q))$. Table 3 summarizes the known results of the codegree densities of projective geometries. It seems that the determination of $\gamma(PG_m(4^k))$ is quite hard. In [8, 11], the authors proved that $\frac{1}{3} \le \gamma(PG_2(4)) < \frac{1}{2}$, and the lower bound is based on the classification of the blocking sets of $PG_2(4)$. In general, we do not know the classification of the blocking sets of $PG_m(q)$.

In this paper, we determine some new families of codegree densities of projective geometries. Our main technique is employing the moment identity of a subset of $PG_m(q)$ (Lemma 2.1). For $m = 4$, we can get similar equations, such as (7) and (8), but there are many solutions since there are more variables. It would be interesting to determine $\gamma(PG_m(q))$ for $m \ge 4$.

Table 3
Known results of $\gamma(PG_m(q))$.

m \ q	2	4	Odd q	2^{2k+1}, $k \geq 1$	2^{2k}, $k \geq 2$
2	$\frac{1}{2}$	$[\frac{1}{3}, \frac{1}{2})$	$\frac{1}{2}$	$\frac{1}{2}$	$[0, \frac{1}{2}]$
3	$\frac{2}{3}$	$[\frac{1}{2}, \frac{2}{3}]$	$\frac{2}{3}$	$\frac{2}{3}$	$[0, \frac{2}{3}]$
4	$[\frac{2}{3}, \frac{3}{4}]$	$[\frac{1}{2}, \frac{3}{4}]$	$[\frac{2}{3}, \frac{3}{4}]$	$[\frac{2}{3}, \frac{3}{4}]$	$[0, \frac{3}{4}]$
$m \geq 5$	$[\frac{3}{4}, 1 - \frac{1}{m}]$	$[\frac{1}{2}, 1 - \frac{1}{m}]$	$[\frac{2}{3}, 1 - \frac{1}{m}]$	$[\frac{2}{3}, 1 - \frac{1}{m}]$	$[0, 1 - \frac{1}{m}]$

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