

COLOR ISOMORPHIC EVEN CYCLES AND A RELATED RAMSEY PROBLEM*

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Abstract. In this paper, we first study a new extremal problem recently posed by Conlon and Tyomkyn [*Repeated Patterns in Proper Colourings*, preprint, <https://arxiv.org/abs/2002.00921> (2020)]. Given a graph H and an integer $k \geq 2$, let $f_k(n, H)$ be the smallest number of colors c such that there exists a proper edge coloring of the complete graph K_n with c colors containing no k vertex-disjoint color-isomorphic copies of H . Using algebraic properties of polynomials over finite fields, we give an explicit proper edge coloring of K_n and show that $f_k(n, C_4) = \Theta(n)$ when $k \geq 3$ and $n \rightarrow \infty$. The methods we used in the edge coloring may be of some independent interest. We also consider a related generalized Ramsey problem. For given graphs G and H , let $r(G, H, q)$ be the minimum number of edge colors (not necessarily proper) of G , such that the edges of every copy of $H \subseteq G$ together receive at least q distinct colors. Establishing the relation to the Turán number of specified bipartite graphs, we obtain some general lower bounds for $r(K_{n,n}, K_{s,t}, q)$ with a broad range of q .

Key words. generalized Ramsey number, edge coloring, algebraic construction

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1. Introduction. The problems of finding rainbow structures in proper edge colorings of complete graphs have been widely studied in recent years. For example, in a recent breakthrough by Montgomery, Pokrovskiy, and Sudakov [24], they confirmed the famous Ringel’s conjecture, one of whose statements involves finding a rainbow copy of any tree with n edges in a particular proper edge coloring of K_{2n+1} . There have also been many papers written on finding large or spanning structures in proper edge colorings; see, e.g., [9, 23, 25, 27]. Very recently, Conlon and Tyomkyn [8] studied a new Ramsey type problem, which aims to find two or more vertex-disjoint color-isomorphic copies of some given graph in proper edge colorings of complete graphs.

For $k, n \geq 2$ and a fixed graph H , define $f_k(n, H)$ to be the smallest integer c such that there exists a proper edge coloring of K_n with c colors containing no k vertex-disjoint color-isomorphic repeats of H . One may ask the following natural question.

PROBLEM 1.1. *Given $k \geq 2$ and a fixed graph H , determine the order of growth of $f_k(n, H)$ as $n \rightarrow \infty$.*

In [8], Conlon and Tyomkyn studied this problem systematically. They first made many useful observations on the properties of $f_k(n, H)$. For instance, $f_k(n, H)$

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is monotone increasing in n , but decreasing in k . Moreover, $f_k(n, H)$ is monotone decreasing in H with respect to taking a subgraph, i.e., $f_k(n, H) \leq f_k(n, H')$ when H' is a subgraph of H . Also, since every proper coloring of K_n uses at least $n - 1$ colors, then $n - 1 \leq f_k(n, H) \leq \binom{n}{2}$. Using the Lovász local lemma and Bukh's random algebraic method [3], they proved the following results.

THEOREM 1.2 (see [8]). *The following hold.*

(i) *For any graph H with v vertices and e edges,*

$$f_k(n, H) = O(\max\{n, n^{\frac{kv-2}{(k-1)^e}}\}).$$

(ii) *For every graph H containing a cycle, there exists $k = k(H)$ such that*

$$f_k(n, H) = \Theta(n).$$

Conlon and Tyomkyn also suggested studying $f_k(n, H)$ when H is an even cycle. Theorem 1.2 implies $f_k(n, C_4) = O(n^{\frac{2k-1}{2k-2}})$ (using the Lovász local lemma), and there is an integer k such that $f_k(n, C_4) = \Theta(n)$ (using the random algebraic method). The constant k obtained by the random algebraic method is likely very large due to the Lang–Weil bound [22], and they asked whether $f_2(n, C_4) = \Theta(n)$.

Our first result in this paper studies $f_k(n, C_4)$. We try to estimate the smallest integer k such that $f_k(n, C_4) = \Theta(n)$, and we give the following result via an algebraic construction.

THEOREM 1.3. $f_3(n, C_4) = \Theta(n)$.

This result improves the best known upper bound $O(n^{\frac{5}{4}})$ obtained by Theorem 1.2(i), and it perhaps gives some evidence that $f_2(n, C_4)$ is also of order $\Theta(n)$.

As the authors mentioned in [8], the problem of studying $f_k(n, H)$ was motivated by a generalized Ramsey problem raised by Krueger [21]. Our next result in this paper studies this generalized Ramsey problem. The classical graph Ramsey problem asks for the minimum number n such that every k -coloring of the edges of K_n forces a monochromatic copy of K_p . By fixing n , the inverse problem asks for the minimum k such that there exists an edge coloring of K_n with k colors, and each copy of K_p receives at least q colors. For general graphs G and H , the generalized Ramsey number $r(G, H, q)$ denotes the minimum number of edge colors of G , such that the edges of every copy of $H \subseteq G$ together receive at least q distinct colors. This function was first studied by Elekes, Erdős, and Füredi (see section 9 of [10]). Later, Erdős and Gyárfás [11] systematically studied the function $r(K_n, K_p, q)$ and showed many improved results. After that, a number of wonderful papers [1, 6, 13, 15, 26, 28] studied this problem, and obtained lots of interesting results in this fashion. In recent years, some questions about distinct distances and difference sets with similar flavors have also been studied in [13, 14, 26].

In this paper, we are interested in the bipartite version of generalized Ramsey number $r(K_{n,n}, K_{s,t}, q)$, which has been studied in [1, 5, 29]. In particular, Axenovich, Füredi, and Mubayi [1] obtained a series of improved results via many different methods such as the Lovász local lemma and algebraic methods. We list some results of [1] in Tables 1 and 2.

In our studies of $r(K_{n,n}, K_{s,t}, q)$, we will always assume that s, t , and q are fixed integers and $n \rightarrow \infty$. Axenovich, Füredi, and Mubayi [1] determined the linear and quadratic thresholds of the function $r(K_{n,n}, K_{s,s}, q)$. More precisely, they determined the smallest integers $q_1(s)$ and $q_2(s)$, where $q_1(s) = s^2 - 2s + 3$ and $q_2(s) = s^2 - s + 2$,

TABLE 1
 $r(K_{n,n}, K_{s,s}, q)$ with $s = 2, 3$.

q	$r(K_{n,n}, K_{2,2}, q)$	$r(K_{n,n}, K_{3,3}, q)$
2	$(1 + o(1))\sqrt{n}$	$(1 + o(1))n^{\frac{1}{3}}$
3	$> \lfloor \frac{2n}{3} \rfloor; \leq n - 1$	$O(n^{\frac{4}{3}})$
4	n^2	$O(n^{\frac{2}{3}})$
5	n^2	$O(n^{\frac{4}{5}})$
6	n^2	$\Theta(n)$
7	n^2	$\Omega(n); O(n^{\frac{4}{3}})$
8	n^2	$\lceil \frac{n}{2} \lceil \frac{3n}{2} \rceil \rceil$
9	n^2	n^2

TABLE 2
 $r(K_{n,n}, K_{s,s}, q)$ with $s \geq 4$.

q	$r(K_{n,n}, K_{s,s}, q)$
2	$\Omega(n^{\frac{1}{s}})$
$s^2 - 2s + 2$	$O(n^{1 - \frac{1}{2s-1}})$
$s^2 - 2s + 3$	$\Theta(n)$
$s^2 - s + 1$	$O(n^{2 - \frac{2}{s}})$
$s^2 - s + 2$	$\geq C_s(n^2 - n); < (1 - c_s)n^2$
$s^2 - \lfloor \frac{2s-1}{3} \rfloor + 1$	$> n^2 - 2\lfloor \frac{s-2}{3} \rfloor(n-1)$
$s^2 - \lfloor \frac{s}{2} \rfloor + 1$	$n^2 - \lfloor \frac{s}{2} \rfloor + 1$
s^2	n^2

such that $r(K_{n,n}, K_{s,s}, q_1(s)) = \Theta(n)$ and $r(K_{n,n}, K_{s,s}, q_2(s)) = \Omega(n^2)$. Up to now, nothing has been shown about $r(K_{n,n}, K_{s,s}, q)$ beyond the trivial lower bound $\Omega(n)$ when $s^2 - 2s + 3 < q < s^2 - s + 2$.

Our next results give some general lower bounds for $r(K_{n,n}, K_{s,t}, q)$ with a broad range of q . Recall that $\text{ex}(n, H)$ denotes the maximum number of edges in an H -free graph G with n vertices.

THEOREM 1.4. *For given integers $t \geq s \geq 4$, we have*

$$r(K_{n,n}, K_{s,t}, st - e(H) + 1) = \Omega\left(\frac{n^4}{\text{ex}(n^2, H)}\right),$$

where H is a bipartite graph with bipartition $H = H_1 \cup H_2$ such that $|H_1| \leq \lfloor \frac{s}{2} \rfloor$ and $|H_2| \leq \lfloor \frac{t}{2} \rfloor$.

When $s = t$ is even, let H be the even cycle of length s . Using the upper bound $\text{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$ by Bondy and Simonovits [2], we obtain the following corollary.

COROLLARY 1.5. *When $s \geq 4$ is an even integer, we have*

$$r(K_{n,n}, K_{s,s}, s^2 - s + 1) = \Omega(n^{2 - \frac{4}{s}}).$$

As we see in Table 2, our lower bound is not far from the best known upper bound $r(K_{n,n}, K_{s,s}, s^2 - s + 1) = O(n^{2 - \frac{2}{s}})$, and Corollary 1.5 gives the first non-trivial lower bound when $s > 4$ is an even integer and $q = s^2 - s + 1$. Note that the

choice of H in Theorem 1.4 is flexible, which leads to a broad range of q in function $r(K_{n,n}, K_{s,t}, q)$. Let $q_1(s, t) = st - s - t + 3$ and $q_2(s, t) = st - \frac{s+t}{2} + 2$. The results in [21] imply $r(K_{n,n}, K_{s,t}, q_1(s, t)) = \Theta(n)$ and $r(K_{n,n}, K_{s,t}, q_2(s, t)) = \Theta(n^2)$. By choosing different graphs H satisfying the conditions in Theorem 1.4, as corollaries, we are able to get superlinear lower bounds for $r(K_{n,n}, K_{s,t}, q)$ with many distinct parameters q .

We give two corollaries in this fashion for the asymmetric version, that is, when $t > s$. The corollaries are obtained by choosing H as a 1-subdivision of complete graphs and a 1-subdivision of complete bipartite graphs, respectively. We utilize known extremal numbers of the 1-subdivision of complete graphs $\text{ex}(n, \text{sub}(K_t)) = O(n^{\frac{3}{2} - \frac{1}{4t-6}})$, and the 1-subdivision of complete bipartite graphs $\text{ex}(n, \text{sub}(K_{s,t})) = O(n^{\frac{3}{2} - \frac{1}{2s}})$ (see [7, 17]).

COROLLARY 1.6. *When $s \geq 6$ is an even integer and $t = 2\binom{s/2}{2}$, we have*

$$r(K_{n,n}, K_{s,t}, st - t + 1) = \Omega(n^{1 + \frac{1}{s-3}}).$$

COROLLARY 1.7. *When $s \geq 8$ is an even integer and $t = \frac{s^2}{8}$, we have*

$$r(K_{n,n}, K_{s,t}, st - t + 1) = \Omega(n^{1 + \frac{4}{s}}).$$

Also, there are not many lower bounds for $r(K_{n,n}, K_{s,t}, q)$ that have been found when $q < q_1(s, t) = st - s - t + 3$. The only known case is when $q = 2$, $r(K_{n,n}, K_{s,s}, 2) = \Omega(n^{\frac{1}{s}})$ [1]. We obtain some new sublinear lower bounds by using Theorem 1.4 and the famous Kövari–Sós–Turán bound [20] $\text{ex}(n, K_{m,\ell}) = O(n^{2 - \frac{1}{m}})$ with $m \leq \ell$.

COROLLARY 1.8. *When $t \geq s \geq 4$, letting $m \leq \frac{s}{2}$ and $\ell \leq \frac{t}{2}$ be integers with $m \leq \ell$, we have*

$$r(K_{n,n}, K_{s,t}, st - m\ell + 1) = \Omega(n^{\frac{2}{m}}).$$

In particular, setting $s = t$ and $m = \ell = \frac{s}{2}$, we have

$$r\left(K_{n,n}, K_{s,s}, \frac{3s^2}{4} + 1\right) = \Omega(n^{\frac{4}{s}}).$$

The paper is organized as follows. In section 2, we prove Theorem 1.3 by giving an algebraic construction. In section 3, we prove Theorem 1.4. Finally we conclude with some remarks and further questions in section 4.

2. Even cycle C_4 . In this section, we are going to prove Theorem 1.3. The proof goes as follows. We first choose a field \mathcal{K} , and construct a map $\pi : V \rightarrow \mathcal{K}$, where $V = V(K_n)$. Then we choose a symmetric polynomial $P \in \mathcal{K}[x, y]$, and color the edge ab by $P(\pi(a), \pi(b))$. We aim to show that under this construction, the edge-coloring we obtained has bounded maximum degree in each color class (thus by a standard application of Vizing's theorem, we are able to get a proper edge coloring), the image¹ $|P(\pi(V), \pi(V))| = O(n)$, and we cannot find too many color isomorphic copies of C_4 in this coloring.

One may choose \mathcal{K} as a field of characteristic 0. Thus, by the symmetric Elekes–Ronyai theorem [19], our symmetric polynomial $P(x, y)$ has the form $f(u(x) + u(y))$ or $f(u(x)u(y))$, where f, u are some one variable polynomials in $\mathcal{K}[x]$. However, constant

¹Given $A, B \subseteq \mathcal{K}$, let $P(A, B)$ denote $\{P(a, b) : a \in A, b \in B\}$.

many color isomorphic copies of C_4 in this case would imply the set $u(\pi(V))$ has low additive or multiplicative energy, and this gives us that the image $|P(\pi(V), \pi(V))|$ is $\Theta(\pi(V)^2)$, providing $\pi(V) = O(n^{\frac{1}{2}})$. Hence the maximum degree of the color class being $\Omega(n^{\frac{1}{2}})$, we will no longer have a proper edge coloring.

We fix this problem by choosing \mathcal{K} to be a finite field \mathbb{F}_p , and we choose our polynomial P to be an expanding polynomial over \mathbb{R} . This means, if we view our polynomial as an element in $\mathbb{R}[x, y]$, and $\pi : V \rightarrow \mathbb{R}$, by taking an injection π , the edge coloring is proper and we will not have many color isomorphic copies of C_4 , but the image $|P(\pi(V), \pi(V))|$ is quadratic in n . Now, by taking $p = O(n)$, mapping everything down to \mathbb{F}_p will help us to decrease $|P(\pi(V), \pi(V))|$ to $O(n)$. By choosing the polynomial $P(x, y)$, the vertex map π , and the character p carefully, and using the ideas from the properties of the resultants of polynomials, we manage to show that we can still get a proper coloring, and taking this projection down to \mathbb{F}_p will not create too many color isomorphic copies of C_4 .

Proof of Theorem 1.3. Let $p \equiv 5 \pmod{6}$ be a sufficiently large prime and

$$A = \left\{ 1, 2, \dots, \left\lfloor \frac{p-1}{3} \right\rfloor \right\}$$

be a subset of \mathbb{F}_p .

We remark that the purpose of choosing $p \equiv 5 \pmod{6}$ is to make -3 a non-residue modulo p . This follows from the law of quadratic reciprocity, which states that if r and s are odd primes, and we define r^* to be r if $r \equiv 1 \pmod{4}$ or $-r$ if $r \equiv 3 \pmod{4}$, then s is a quadratic residue modulo r if and only if r^* is a quadratic residue modulo s . Applying this with $r = 3$ and $s = p$, we see that -3 is a quadratic residue modulo p if and only if p is a quadratic residue modulo 3 , i.e., $p \equiv 1 \pmod{3}$.

Now, let G be a complete graph with vertex set A . Let $P(x, y) = x^2 + xy + y^2$ be an element in $\mathbb{F}_p[x, y]$. Let $P^*(A, A)$ be the restricted image, that is

$$P^*(A, A) = \{P(a, b) : a, b \in A, a \neq b\}.$$

Define the edge coloring

$$\chi : E(G) \rightarrow P^*(A, A),$$

such that for every $x, y \in A$ with $x \neq y$, we assign color $P(x, y)$ to the edge xy . Note that $x^2 + xy + y^2 = (x + \frac{y}{2})^2 + \frac{3}{4}y^2$, and -3 is a nonquadratic residue modulo p , then $x^2 + xy + y^2 \neq 0$. Hence $P^*(A, A) \subseteq \mathbb{F}_p^*$.

Next, we claim that the edge coloring χ is proper. Suppose there is $a \in \mathbb{F}_p^*$, such that two edges xy and xz are assigned to the same color a . That is, we have $x^2 + xy + y^2 = a$ and $x^2 + xz + z^2 = a$. Hence $(y - z)(x + y + z) = 0$. By the way we construct the vertex set A , $x + y + z \neq 0$. This implies $y = z$. Therefore, two distinct edges xy and xz cannot be assigned the same color, then χ is proper.

Let $a, b, c, d \in \mathbb{F}_p^*$, and $a \neq b, b \neq c, c \neq d, d \neq a$. Assume the colors a, b, c, d are incident to a four-cycle $x, y, z, w \in A$. Then we have

- (2.1) $x^2 + xy + y^2 = a,$
- (2.2) $y^2 + yz + z^2 = b,$
- (2.3) $z^2 + zw + w^2 = c,$
- (2.4) $w^2 + wx + x^2 = d.$

From (2.1) and (2.2), we obtain

$$(2.5) \quad y = \frac{a-b}{x-z} - x - z.$$

Similarly, from (2.3) and (2.4), we obtain

$$(2.6) \quad w = \frac{c-d}{z-x} - z - x.$$

Substituting (2.5) and (2.6) into (2.1) and (2.4), respectively, we have

$$(2.7) \quad x^2 + x\left(\frac{a-b}{x-z} - x - z\right) + \left(\frac{a-b}{x-z} - x - z\right)^2 = a,$$

$$(2.8) \quad x^2 + x\left(\frac{c-d}{z-x} - z - x\right) + \left(\frac{c-d}{z-x} - z - x\right)^2 = d.$$

Making the change of variables $u \mapsto x+z$ and $v \mapsto x-z$ in (2.7) and (2.8), we get $v \neq 0$, and

$$(2.9) \quad v^4 + 3u^2v^2 - 6(a-b)uv - 2(a+b)v^2 + 4(a-b)^2 = 0,$$

$$(2.10) \quad v^4 + 3u^2v^2 - 6(c-d)uv - 2(c+d)v^2 + 4(c-d)^2 = 0.$$

Assume that $a-b = c-d$. Then from (2.9) and (2.10), we have $a+b = c+d$, hence $a = c, b = d$. By (2.5) and (2.6), we have $y+w+2x+2z = 0$ and, by the symmetry, we can also obtain $x+z+2y+2w = 0$. Therefore, $x+z = y+w = 0$, which contradicts the construction of A .

Now we have that $a-b \neq c-d$. From (2.9) and (2.10) we can obtain

$$(2.11) \quad uv = \frac{(a+b-c-d)v^2}{3(c-d-a+b)} + \frac{2}{3}(c-d+a-b).$$

Substituting (2.11) into (2.9), and then multiplying by $3(c-d-a+b)^2$, we obtain

$$(2.12) \quad k_2v^4 + k_1v^2 + k_0 = 0,$$

where

$$k_2 = 4(a^2 - ab - 2ac + ad + b^2 + bc - 2bd + c^2 - cd + d^2),$$

$$k_1 = 4(a-b-c+d)(a^2 - 2ac - 3ad - b^2 + 3bc + 2bd + c^2 - d^2),$$

$$k_0 = 4(a-b-c+d)^2(a^2 - 2ab - ac + ad + b^2 + bc - bd + c^2 - 2cd + d^2).$$

Assume first that $k_2 = k_1 = k_0 = 0$. Since $a-b-c+d \neq 0$, we have

$$(2.13) \quad a^2 - ab - 2ac + ad + b^2 + bc - 2bd + c^2 - cd + d^2 = 0,$$

$$(2.14) \quad a^2 - 2ab - ac + ad + b^2 + bc - bd + c^2 - 2cd + d^2 = 0.$$

From (2.13) and (2.14), we obtain $(a-d)(b-c) = 0$, which contradicts the choice of a, b, c, d .

Recall that $v = x-z$ and $x, z \in A$. Thus if at least one of k_i ($i = 0, 1, 2$) is not 0, then there are at most 4 solutions for v in $(A-A) \setminus \{0\}$. Also, if the number of solutions for v is at least 3, then there exist two solutions v_1, v_2 such that $v_1 + v_2 = 0$. In this

case, let $x_1, z_1, u_1 (x_2, z_2, u_2)$ be the corresponding solutions with respect to $v_1 (v_2, respectively)$. Then $v_1^2 = v_2^2$ and, by (2.11), we have $u_1v_1 = u_2v_2$. Hence $u_1 = -u_2$. Note that since $v_i = x_i - z_i$ and $u_i = x_i + z_i$, we have $x_1 - z_1 = -x_2 + z_2$ and $x_1 + z_1 = -x_2 - z_2$. Then we can get $x_1 + x_2 = 0$, which contradicts the construction of A .

Hence there are at most 2 solutions for v in $(A - A) \setminus \{0\}$. For any fixed v , by (2.11), there is a unique solution for u . Note that $x = \frac{u+v}{2}$ and $z = \frac{u-v}{2}$ and, by (2.5) and (2.6), y and w are uniquely determined by x and z , hence there are at most 2 solutions for (x, y, z, w) . Therefore, the number of copies of four-cycles with edge colors a, b, c, d is at most two. \square

3. Lower bounds for $r(K_{n,n}, K_{s,t}, q)$. In this section, we are going to prove Theorem 1.4. The ideas used in this proof are mainly inspired by the recent work of Conlon and Tyomkyn [8]. It will often be helpful to think of $r(K_{n,n}, K_{s,t}, q)$ in terms of repeated colors. Let \mathcal{C} be the collection of colors, and let $\chi : E(G) \rightarrow \mathcal{C}$ be an edge coloring of graph G . Let H be a subgraph of G . If a color $c \in \mathcal{C}$ appears on exactly r_c edges in $\chi(E(H))$, then we say such a color c is repeated $r_c - 1$ times in H . We say H has r repeats if $r = \sum_{c \in \chi(E(H))} (r_c - 1)$, where every color $c \in \chi(E(H))$ is repeated $r_c - 1$ times, and the sum is taking over all colors in $\chi(E(H))$ (hence $r_c \geq 1$).

Proof of Theorem 1.4. Suppose that n is sufficiently large, and let

$$\chi : E(K_{n,n}) \rightarrow \mathcal{C}$$

be an edge coloring of $K_{n,n}$, where \mathcal{C} is the collection of colors. Suppose $K_{n,n}$ has the vertex bipartition $A \cup B$. We label the vertices in A and B , respectively, such that $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ with $a_1 < a_2 < \dots < a_n$ and $b_1 < b_2 < \dots < b_n$.

Now we construct the auxiliary graph F as follows. F is a bipartite graph with vertex set $U \cup W$, such that $U = \binom{A}{2}$ and $W = \binom{B}{2}$. Thus $|U| = |W| = \binom{n}{2}$. Moreover, we require the elements in U to have the form (a_i, a_j) with $a_i > a_j$, and elements in W to have the form (b_k, b_ℓ) with $b_k > b_\ell$. For every $(a_i, a_j) \in U$ and $(b_k, b_\ell) \in W$, (a_i, a_j) and (b_k, b_ℓ) are adjacent in F if $\chi(a_i b_k) = \chi(a_j b_\ell)$ in the edge coloring of $K_{n,n}$. Given $c \in \mathcal{C}$, letting e_c be the number of edges of color c in the image of χ , we have

$$e(F) = \sum_{c \in \mathcal{C}} \binom{e_c}{2} \geq \frac{(\sum_{c \in \mathcal{C}} e_c)^2}{4|\mathcal{C}|} = \frac{n^4}{4|\mathcal{C}|}.$$

Hence $|\mathcal{C}| \geq \frac{n^4}{4e(F)}$.

Next, we are going to bound $e(F)$ and, hence, get a lower bound on $|\mathcal{C}|$. Let H be a bipartite graph, with vertex set $H_1 \cup H_2$, such that $|H_1| \leq \lfloor \frac{s}{2} \rfloor$ and $|H_2| \leq \lfloor \frac{t}{2} \rfloor$. Suppose $e(F) \geq \text{ex}(|V(F)|, H)$, then F contains a copy of H . Observe that, by the definition of the auxiliary graph F , every edge of the copy H in F will contribute exactly one repeat in the edge coloring χ of $K_{n,n}$. Thus, there are at least $e(H)$ repeats in χ . Moreover, all these $e(H)$ repeats span at most $2|H_1|$ vertices in A , and at most $2|H_2|$ vertices in B . Thus by the upper bound on $|H_1|$ and $|H_2|$, we are able to find a copy of $K_{s,t}$ in $K_{n,n}$ such that $|\chi(E(K_{s,t}))|$ is at most $st - e(H)$. Therefore, if the image of χ does not contain a $K_{s,t}$ with less than $st - e(H) + 1$ colors, we have $e(F) \leq \text{ex}(|V(F)|, H)$, finishing the proof. \square

4. Concluding remarks. Although the proof of Theorem 1.3 only requires an elementary computation, it is motivated by considering the resultants of polynomials. Let us first recall the definition of the resultant of polynomials over $\mathcal{K}[x]$.

CONJECTURE 4.4. For any $\ell \geq 3$, $f_2(n, C_{2\ell}) = \Omega(n^{2-\frac{2}{\ell}})$.

Conlon and Tyomkyn [8] verified this conjecture when $\ell = 3$. The proof relies on the upper bound for $\text{ex}(n, \theta_{\ell,t})$ [12] and the observation that the endpoints of theta graph $\theta_{\ell,t}$ cannot be in the same part when ℓ is odd (this key observation is also useful in [4]).

Note added. Very recently, Janzer [18] developed a method for finding suitable cycles of given length and then proved Conjecture 4.4 in a more general form.

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