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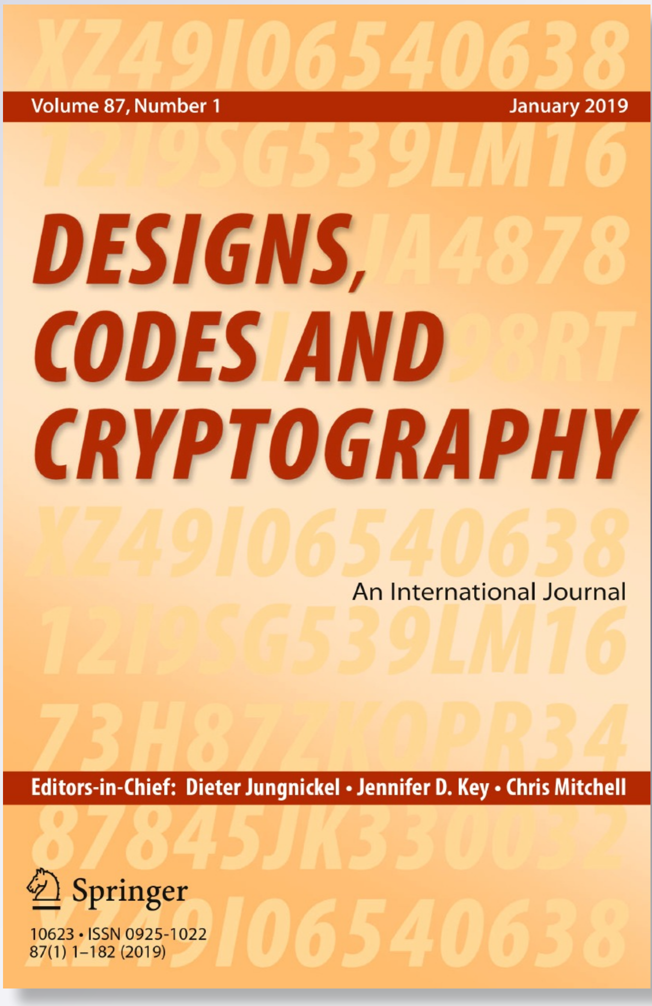


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# Constructions of optimal Ferrers diagram rank metric codes

**Tao Zhang**[1] · **Gennian Ge**[1]

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**Abstract** Subspace codes and constant dimension codes have become a widely investigated research topic due to their significance to error control in random linear network coding. Rank metric codes in Ferrers diagrams can be used to construct good subspace codes and constant dimension codes. In this paper, three constructions of Ferrers diagram rank metric codes are presented. The first two constructions are based on subcodes of maximum rank distance codes, and the last one generates new codes from known Ferrers diagram rank metric codes. Each of these constructions produces optimal codes with different diagrams and parameters for which no optimal construction was known before.

**Keywords** Ferrers diagram · Rank metric code · Gabidulin code · Subspace code · Constant dimension code

**Mathematics Subject Classification** 15A03 · 15A99 · 15B99

## 1 Introduction

In 2008, Koetter and Kschischang [10] presented an application of subspace codes for error correction in random network coding. After this work, the theory of subspace codes has developed rapidly. Constructions and bounds for subspace codes and constant dimension codes can be found in [1,4,6,8–12,15–17,19].

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Note that the codes obtained by lifting the maximum rank distance (MRD) codes are asymptotically optimal constant dimension codes [16]. However, these codes are not optimal. It is of interest to construct constant dimension codes and subspace codes with large size. Several constructions of constant dimension codes and subspace codes with large cardinality can be found in [4–6,10,17].

In [4], the multilevel construction was presented. It is a generalization of the lifted MRD codes and provides constant dimension codes with the largest known cardinality. The multilevel construction is based on the union of several rank metric codes having a given shape of their codewords, namely, Ferrers diagram rank metric codes. Further, improvements of the multilevel construction considering additionally the so-called pending dots of the Ferrers diagram for the construction of constant dimension codes were given in [13,14,18].

A Ferrers diagram is an array of dots and empty entries, and a Ferrers diagram rank metric code is a set of matrices where only the entries with dots in the Ferrers diagram are allowed to be nonzero. An upper bound on the cardinality of such Ferrers diagram rank metric codes was given in [4] and a specific construction of such Ferrers diagram rank metric codes was given in the same paper. Later, Etzion et al. [3] presented four new constructions of optimal Ferrers diagram rank metric codes.

The main goal of this paper is to continue this research. We will give three constructions of Ferrers diagram rank metric codes. In particular, we obtain optimal codes for different diagrams and parameters for which no optimal construction was known before. This paper is organized as follows. In Sect. 2, we give some basic results about rank metric codes and Ferrers diagram rank metric codes. In Sect. 3, we present three constructions of optimal Ferrers diagram rank metric codes. Section 4 concludes the paper.

## 2 Preliminaries

Throughout this paper, the following notations will be fixed.

- Let $q$ be a power of a prime, $\mathbb{F}_q$ be the finite field of order $q$.
- Let $\mathbb{F}_q^{m\times n}$ be the set of all $m \times n$ matrices over $\mathbb{F}_q$, $\mathbb{F}_q^n$ be the set of all row vectors of length $n$ over $\mathbb{F}_q$.
- The rows and columns of an $m \times n$ matrix $A$ will be indexed by $0, \ldots, m-1$ and $0, \ldots, n-1$, respectively.
- $[a, b] = \{i : a \leq i \leq b, i \in \mathbb{Z}\}$.

Without loss of generality, we assume that $n \leq m$.

### 2.1 Rank metric codes

The rank distance on the $\mathbb{F}_q^{m\times n}$ is defined by

$$d(A, B) = \mathrm{rank}_q(A - B) \text{ for } A, B \in \mathbb{F}_q^{m\times n},$$

where $\mathrm{rank}_q(C)$ stands for the rank of $C$ over the field $\mathbb{F}_q$. A subset $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ is called a rank metric code. The minimum distance of $\mathcal{C}$ is

$$d(\mathcal{C}) = \min_{A,B\in\mathcal{C},A\neq B}\{d(A, B)\}.$$

When $\mathcal{C}$ is an $\mathbb{F}_q$ linear subspace of $\mathbb{F}_q^{m\times n}$, we say that $\mathcal{C}$ is an $\mathbb{F}_q$ linear code and its dimension $\dim_q(\mathcal{C})$ is defined to be the dimension of $\mathcal{C}$.

Let $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ and $d(\mathcal{C}) = d$, it is well known that

$$\#\mathcal{C} \le q^{\max\{m,n\}(\min\{m,n\}-d+1)},$$

which is the Singleton bound for the rank metric code, see [2]. When the equality holds, we call $\mathcal{C}$ a maximum rank distance (MRD) code. There are some constructions of MRD codes, the first and most famous family is due to Delsarte [2] and Gabidulin [7] who found it independently. Since their works, we know that linear MRD codes exist for any set of parameters.

If we choose a fixed basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, then any vector $c \in \mathbb{F}_{q^m}^n$ may be regarded as a matrix in $\mathbb{F}_q^{m\times n}$. Based on this fact, the Gabidulin codes can be constructed as follows.

**Definition 2.1** (*Gabidulin Code* [2,7]) A linear $G[m \times n, \delta]_q^R$ MRD code, in vector representation over $\mathbb{F}_{q^m}^n$, of dimension $m(n-\delta+1)$ over $\mathbb{F}_q$ and minimum rank distance $\delta$ is defined by its $(n-\delta+1)\times n$ generator matrix $G$:

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-1} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-1}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-1}^{[n-\delta]} \end{pmatrix},$$

where $a^{[i]} = a^{q^i}$ and $g_0, g_1, \ldots, g_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over $\mathbb{F}_q$.

## 2.2 Rank metric codes in Ferrers diagram

The Ferrers diagrams are defined as follows.

**Definition 2.2** An $m \times n$ Ferrers diagram $\mathcal{F}$ is an $m \times n$ array of dots and empty entries with the following properties:

- all dots are shifted to the right,
- the number of dots in each row is at most the number of dots in the previous row,
- the first row has $n$ dots and the rightmost column has $m$ dots.

The number of dots in $\mathcal{F}$ is denoted by $|\mathcal{F}|$. We also use $\mathbb{F}_q[\mathcal{F}]$ to denote the set of all matrices in $\mathbb{F}_q^{m\times n}$ with nonzero entries only at positions where $\mathcal{F}$ has dots. Clearly, $\mathbb{F}_q[\mathcal{F}]$ is a linear space over $\mathbb{F}_q$ and the dimension of $\mathbb{F}_q[\mathcal{F}]$ over $\mathbb{F}_q$ is $|\mathcal{F}|$. Further, an $m \times n$ Ferrers diagram is called full Ferrers diagram if it has $mn$ dots.

*Example 2.3* The following example shows a $4 \times 4$ Ferrers diagram $\mathcal{F}$ with $|\mathcal{F}| = 10$,

$$\mathcal{F} = \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & & \bullet & \bullet \\ & & & \bullet \end{matrix}.$$

The Ferrers diagram rank metric code is a set of matrices having nonzero entries only at positions where the Ferrers diagram has dots and equipped with the rank distance metric. For a given $m \times n$ Ferrers diagram $\mathcal{F}$, the triple $[\mathcal{F}, k, \delta]_q^R$ denotes a linear Ferrers diagram rank metric code in $\mathbb{F}_q[\mathcal{F}]$ with minimum rank distance $\delta$ and dimension $k$. Since the cardinality of the constant dimension code obtained increases with the dimension of the Ferrers diagram rank metric code, it is natural to ask what is the maximum possible dimension of the Ferrers diagram rank metric code with Ferrers diagram $\mathcal{F}$ and minimum distance $\delta$.

Let $\gamma_i, i \in [0, n-1]$ be the number of dots in $\mathcal{F}$ in the $i$-th column. For given $\mathcal{F}$ and $\delta$, the maximum dimension of an associated rank metric code is denoted by $\dim(\mathcal{F}, \delta)$. An upper bound on $\dim(\mathcal{F}, \delta)$ was given in [4]. This is a Singleton-like bound for Ferrers diagram rank metric codes.

**Theorem 2.4** [4] *Let $v_i$, $i \in [0, \delta - 1]$, denote the number of dots in a Ferrers diagram $\mathcal{F}$ after removing the first $i$ rows and the $\delta - 1 - i$ rightmost columns. Then,*

$$dim(\mathcal{F}, \delta) \leq min_{i \in [0, \delta-1]} v_i.$$

The codes which attain this bound are called *optimal*. It is easy to see that MRD codes attain the upper bound of Theorem 2.4 for full Ferrers diagram. Etzion and Silberstein proposed the following conjecture.

**Conjecture 1** [4] *The upper bound of Theorem 2.4 is attainable for any given set of parameters $\mathbb{F}_q$, $\mathcal{F}$, and $\delta$.*

The following theorems give some known constructions of optimal Ferrers diagram rank metric codes.

**Theorem 2.5** [4] *Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram and assume that each of the $\delta - 1$ rightmost columns has $m$ dots. Then there exists an $[\mathcal{F}, k, \delta]_q^R$ rank metric code attaining the bound from Theorem 2.4 for any $q \geq 2$ and any $\delta$.*

**Theorem 2.6** [3] *Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram and assume that each of the $\delta - 1$ rightmost columns has $n$ dots. Then there exists an $[\mathcal{F}, k, \delta]_q^R$ rank metric code attaining the bound from Theorem 2.4 for any $q \geq 2$ and any $\delta$.*

**Theorem 2.7** [3] *Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram and assume that each of the $\delta - 1$ rightmost columns has at least $n-1$ dots and the rightmost column has $\gamma_{n-1} = m \geq n-1+\gamma_0$ dots. Then there exists an $[\mathcal{F}, k, \delta]_q^R$ rank metric code attaining the bound from Theorem 2.4 for any $q \geq 2$ and any $\delta$.*

*Remark 2.8* In [3], the authors also presented a construction of optimal Ferrers diagram rank metric codes from MDS codes. Moreover, they gave two constructions which combine codes in small diagrams to generate a code in a larger diagram, see [3] for details.

## 3 Constructions

In this section, we give three constructions of Ferrers diagram rank metric codes, of which the first two are based on subcodes of MRD codes and the last one is from known Ferrers diagram rank metric codes. As a preparation, we give the definition of $\psi_m$, which is a map from $\mathbb{F}_{q^m}^n$ to $\mathbb{F}_q^{m \times n}$.

Let $\beta_0, \beta_1, \ldots, \beta_{m-1}$ be an ordered basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. There is a bijective map $\psi_m$ of any vector $a \in \mathbb{F}_{q^m}^n$ on a matrix $A \in \mathbb{F}_q^{m \times n}$, denoted as follows:

$$\psi_m : \mathbb{F}_{q^m}^n \mapsto \mathbb{F}_q^{m \times n}$$
$$a = (a_0, a_1, \ldots, a_{n-1}) \mapsto A,$$

where $A = \psi_m(a) \in \mathbb{F}_q^{m \times n}$ is defined below such that

$$a_j = \sum_{i=0}^{m-1} A_{ij} \beta_i, \ \ j \in [0, n-1].$$

### 3.1 Construction I

We first illustrate our idea with an example. Note that for the parameters of this example, none of the previous constructions gives optimal codes.

*Example 3.1* Let $\mathcal{F}$ be the following $8 \times 8$ Ferrers diagram:

$$\mathcal{F} = \begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \end{array}.$$

For $\delta = 6$, the upper bound gives $\dim(\mathcal{F}, \delta) \leq 6$.

Let $g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7$ be a basis of $\mathbb{F}_{q^8}$ over $\mathbb{F}_q$ with $g_0, g_1, g_2, g_3 \in \mathbb{F}_{q^4}$. Then the code with generator matrix

$$G := \begin{pmatrix} g_0 & g_1 & \cdots & g_7 \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_7^{[1]} \\ g_0^{[2]} & g_1^{[2]} & \cdots & g_7^{[2]} \end{pmatrix}$$

is a linear $G[8 \times 8, 6]_q^R$ MRD code. There exists a matrix $A \in GL_3(\mathbb{F}_{q^4})$ such that

$$AG = \begin{pmatrix} 1 & 0 & 0 & b_{01} & b_{02} & \cdots & b_{05} \\ 0 & 1 & 0 & b_{11} & b_{12} & \cdots & b_{15} \\ 0 & 0 & 1 & b_{21} & b_{22} & \cdots & b_{25} \end{pmatrix}$$

and $b_{01}, b_{11}, b_{21} \in \mathbb{F}_{q^4}$. The matrix $AG$ generates the same code as matrix $G$. The set

$$\widehat{C} = \left\{ u \cdot AG : u = (u_0, u_1, u_2) \in \mathbb{F}_{q^4}^3,\ \psi_8(u_0) = \begin{pmatrix} u_{00} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \psi_8(u_1) = \begin{pmatrix} u_{10} \\ u_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\right.$$
$$\left. \psi_8(u_2) = \begin{pmatrix} u_{20} \\ u_{21} \\ u_{22} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ u_{00}, u_{10}, u_{11}, u_{20}, u_{21}, u_{22} \in \mathbb{F}_q \right\}$$

is a code of cardinality $q^6$. Let $C$ be the matrix representation of each codeword of $\widehat{C}$. Note that $b_{01}, b_{11}, b_{21}, u_0, u_1, u_2 \in \mathbb{F}_{q^4}$, then the fourth element of $u \cdot AG$ is $u_0 b_{01} + u_1 b_{11} + u_2 b_{21} \in$

$\mathbb{F}_{q^4}$. Hence it is easy to see that all the elements of $C$ have the form

$$\begin{pmatrix} u_{00} & u_{10} & u_{20} & u_{30} & u_{40} & u_{50} & u_{60} & u_{70} \\ 0 & u_{11} & u_{21} & u_{31} & u_{41} & u_{51} & u_{61} & u_{71} \\ 0 & 0 & u_{22} & u_{32} & u_{42} & u_{52} & u_{62} & u_{72} \\ 0 & 0 & 0 & u_{33} & u_{43} & u_{53} & u_{63} & u_{73} \\ 0 & 0 & 0 & 0 & u_{44} & u_{54} & u_{64} & u_{74} \\ 0 & 0 & 0 & 0 & u_{45} & u_{55} & u_{65} & u_{75} \\ 0 & 0 & 0 & 0 & u_{46} & u_{56} & u_{66} & u_{76} \\ 0 & 0 & 0 & 0 & u_{47} & u_{57} & u_{67} & u_{77} \end{pmatrix}.$$

Therefore, $C$ is an $[\mathcal{F}, 6, 6]_q^R$ code.

Now we generalize the construction from Example 3.1.

**Theorem 3.2** *Let $1 = t_0 < t_1 < t_2 < \cdots < t_l \leq m$ be integers with $t_1 \mid t_2 \mid \cdots \mid t_l$. Let $n, \delta$ be integers for which $t_{l-1} < n \leq t_l$ and $n - t_1 + 1 < \delta \leq n$. Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram such that*

- $\gamma_{n-\delta} \leq t_1$;
- $\gamma_{n-\delta+1} \geq t_1$;
- $\gamma_{t_i} \geq t_{i+1}$, $i \in [1, l-1]$.

*Then there exists an optimal $[\mathcal{F}, k, \delta]_q^R$ rank metric code attaining the bound from Theorem 2.4 for any $q \geq 2$.*

*Proof* Let $g_0, g_1, \ldots, g_{t_l - 1}$ be a basis of $\mathbb{F}_{q^{t_l}}$ over $\mathbb{F}_q$ with $g_0 = 1$ and $g_{t_i}, \ldots, g_{t_{i+1}-1} \in \mathbb{F}_{q^{t_{i+1}}}$, $0 \leq i \leq l-1$. Then the code with generator matrix

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-1} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-1}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-1}^{[n-\delta]} \end{pmatrix}$$

is a linear $G[t_l \times n, \delta]_q^R$ MRD code. Note that

$$\begin{pmatrix} g_0 & g_1 & \cdots & g_{n-\delta} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-\delta}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-\delta}^{[n-\delta]} \end{pmatrix}$$

is an invertible matrix over $\mathbb{F}_{q^{t_1}}$, then there exists a matrix $A \in GL_{n-\delta+1}(\mathbb{F}_{q^{t_1}})$ such that

$$AG = \begin{pmatrix} 1 \; 0 \cdots 0 & b_{01} & b_{02} & \cdots & b_{0,\delta-1} \\ 0 \; 1 \cdots 0 & b_{11} & b_{12} & \cdots & b_{1,\delta-1} \\ \vdots \; \vdots \; \ddots \; \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \; 0 \cdots 1 & b_{n-\delta,1} & b_{n-\delta,2} & \cdots & b_{n-\delta,\delta-1} \end{pmatrix}$$

and $b_{ij} \in \mathbb{F}_{q^{t_k}}$ for $0 \leq i \leq n-\delta$, $1 \leq j \leq \delta-1$, where $1 \leq k \leq l$, $t_{k-1} < j+n-\delta+1 \leq t_k$. The matrix $AG$ generates the same code as $G$.

Let the code $C$ be the set of all matrices of the following form:

$$C = \left\{ \psi_{t_l}(c) \in \mathbb{F}_q^{t_l \times n} : c = u \cdot AG \in \mathbb{F}_{q^{t_l}}^n, u = (u_0, u_1, \ldots, u_{n-\delta}) \in \mathbb{F}_{q^{t_1}}^{n-\delta+1}, \psi_{t_l}(u_i) = \begin{pmatrix} u_{i,0} \\ \vdots \\ u_{i,\gamma_i - 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, 0 \le i \le n - \delta \right\}.$$

Note that $u \in \mathbb{F}_{q^{t_1}}^{n-\delta+1}$ and $b_{ij} \in \mathbb{F}_{q^{t_k}}$ for $0 \le i \le n-\delta,\ 1 \le j \le \delta-1,\ 1 \le k \le l,\ t_{k-1} < j + n - \delta + 1 \le t_k$, then $(u \cdot AG)_{j+n-\delta} = \sum_{i=0}^{n-\delta} u_i b_{ij} \in \mathbb{F}_{q^{t_k}}$. Since $\gamma_{j+n-\delta} \ge \gamma_{t_{k-1}} \ge t_k$, then the code $C$ is an $[\mathcal{F}, \sum_{i=0}^{n-\delta} \gamma_i, \delta]$ rank metric code. Here we regard $C \subseteq \mathbb{F}_q^{m \times n}$, where the last $m - t_l$ rows of codewords of $C$ are 0.

On the other hand, the upper bound on the code dimension for this type of Ferrers diagrams can be obtained by deleting $\delta - 1$ rightmost columns, and therefore $\dim(\mathcal{F}, \delta) \le \sum_{i=0}^{n-\delta} \gamma_i$. Our construction attains this optimal dimension and has minimum rank distance $\delta$. $\square$

*Remark 3.3* Taking $t_1 = 4, t_2 = m = n = 8, \delta = 6$ in Theorem 3.2, we can obtain Example 3.1.

### 3.2 Construction II

Before we give our second construction, we give an example to illustrate our idea. This construction can be seen as the combination of the above construction and the method in [3]. Note that for the parameters of the following example, none of the previous constructions gives optimal codes.

*Example 3.4* Let $\mathcal{F}$ be the following $9 \times 9$ Ferrers diagram:

$$\mathcal{F} = \begin{array}{ccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & & & \bullet \end{array}.$$

For $\delta = 7$, the upper bound gives $\dim(\mathcal{F}, \delta) \le 6$.

Let $g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7$ be a basis of $\mathbb{F}_{q^8}$ over $\mathbb{F}_q$ with $g_0, g_1, g_2, g_3 \in \mathbb{F}_{q^4}$. Then the code with generator matrix

$$G := \begin{pmatrix} g_0 & g_1 & \cdots & g_7 \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_7^{[1]} \\ g_0^{[2]} & g_1^{[2]} & \cdots & g_7^{[2]} \end{pmatrix}$$

is a linear $G[8\times 8,6]_q^R$ MRD code. There exists a matrix $A\in GL_3(\mathbb{F}_{q^4})$ such that

$$AG=\begin{pmatrix}1 & 0 & 0 & b_{01} & \cdots & b_{05}\\ 0 & 1 & 0 & b_{11} & \cdots & b_{15}\\ 0 & 0 & 1 & b_{21} & \cdots & b_{25}\end{pmatrix}$$

and $b_{01},b_{11},b_{21}\in\mathbb{F}_{q^4}$. The matrix $AG$ generates the same code as matrix $G$.

Further, let

$$\widehat{G}=\begin{pmatrix}1 & 0 & 0 & b_{01} & \cdots & b_{05} & 0\\ 0 & 1 & 0 & b_{11} & \cdots & b_{15} & b_{16}\\ 0 & 0 & 1 & b_{21} & \cdots & b_{25} & b_{26}\end{pmatrix}.$$

Since $\operatorname{rank}\left(\psi_8\left(b\cdot\begin{pmatrix}1 & 0 & 0 & b_{01} & \cdots & b_{05}\\ 0 & 1 & 0 & b_{11} & \cdots & b_{15}\\ 0 & 0 & 1 & b_{21} & \cdots & b_{25}\end{pmatrix}\right)\right)\geq 6$ for any nonzero vector $b\in\mathbb{F}_{q^8}^3$, then $\operatorname{rank}\left(\psi_8\left(a\cdot\begin{pmatrix}1 & 0 & b_{11} & \cdots & b_{15}\\ 0 & 1 & b_{21} & \cdots & b_{25}\end{pmatrix}\right)\right)\geq 6$ for any nonzero vector $a\in\mathbb{F}_{q^8}^2$ (take $b=(0,a)$). By [3, Lemma 5], we can choose $b_{16},b_{26}\in\mathbb{F}_{q^8}$ such that

$$\operatorname{rank}\left(\psi_8\left(a\cdot\begin{pmatrix}1 & 0 & b_{11} & \cdots & b_{15} & b_{16}\\ 0 & 1 & b_{21} & \cdots & b_{25} & b_{26}\end{pmatrix}\right)\right)\geq 7.$$

Then the set

$$\widehat{C}=\left\{u\cdot\widehat{G}: u=(u_0,u_1,u_2)\in\mathbb{F}_{q^4}^3,\ \psi_8(u_0)=\begin{pmatrix}u_{00}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix},\ \psi_8(u_1)=\begin{pmatrix}u_{10}\\u_{11}\\0\\0\\0\\0\\0\\0\end{pmatrix},\right.$$
$$\left.\psi_8(u_2)=\begin{pmatrix}u_{20}\\u_{21}\\u_{22}\\0\\0\\0\\0\\0\end{pmatrix},\ u_{00},u_{10},u_{11},u_{20},u_{21},u_{22}\in\mathbb{F}_q\right\}$$

is a code of cardinality $q^6$. The matrix representation of a codeword $c\in\widehat{C}$ is

$$\begin{pmatrix}u_{00} & u_{10} & u_{20} & u_{30} & u_{40} & u_{50} & u_{60} & u_{70} & u_{80}\\ 0 & u_{11} & u_{21} & u_{31} & u_{41} & u_{51} & u_{61} & u_{71} & u_{81}\\ 0 & 0 & u_{22} & u_{32} & u_{42} & u_{52} & u_{62} & u_{72} & u_{82}\\ 0 & 0 & 0 & u_{33} & u_{43} & u_{53} & u_{63} & u_{73} & u_{83}\\ 0 & 0 & 0 & 0 & u_{44} & u_{54} & u_{64} & u_{74} & u_{84}\\ 0 & 0 & 0 & 0 & u_{45} & u_{55} & u_{65} & u_{75} & u_{85}\\ 0 & 0 & 0 & 0 & u_{46} & u_{56} & u_{66} & u_{76} & u_{86}\\ 0 & 0 & 0 & 0 & u_{47} & u_{57} & u_{67} & u_{77} & u_{87}\end{pmatrix}.$$

We associate the matrix representation of each codeword of $\widehat{C}$ with the first eight rows of the Ferrers diagram $\mathcal{F}$ and additionally, on the bottom right corner, we place the repetition of $u_{00}$. Let $C$ be the set of all such matrices, i.e.:

$$C = \left\{ \begin{pmatrix} u_{00} & u_{10} & u_{20} & u_{30} & u_{40} & u_{50} & u_{60} & u_{70} & u_{80} \\ 0 & u_{11} & u_{21} & u_{31} & u_{41} & u_{51} & u_{61} & u_{71} & u_{81} \\ 0 & 0 & u_{22} & u_{32} & u_{42} & u_{52} & u_{62} & u_{72} & u_{82} \\ 0 & 0 & 0 & u_{33} & u_{43} & u_{53} & u_{63} & u_{73} & u_{83} \\ 0 & 0 & 0 & 0 & u_{44} & u_{54} & u_{64} & u_{74} & u_{84} \\ 0 & 0 & 0 & 0 & u_{45} & u_{55} & u_{65} & u_{75} & u_{85} \\ 0 & 0 & 0 & 0 & u_{46} & u_{56} & u_{66} & u_{76} & u_{86} \\ 0 & 0 & 0 & 0 & u_{47} & u_{57} & u_{67} & u_{77} & u_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{00} \end{pmatrix} \right\}.$$

It is easy to see that $C$ is an $[\mathcal{F}, k, \delta]_q^R$ code of dimension $k = 6$. It remains to show the minimum rank weight of any nonzero codeword of $C$. We distinguish between two cases:

- $u_{00} \neq 0$. The matrix

$$\begin{pmatrix} u_{00} & u_{10} & u_{20} & u_{30} & u_{40} & u_{50} & u_{60} & u_{70} \\ 0 & u_{11} & u_{21} & u_{31} & u_{41} & u_{51} & u_{61} & u_{71} \\ 0 & 0 & u_{22} & u_{32} & u_{42} & u_{52} & u_{62} & u_{72} \\ 0 & 0 & 0 & u_{33} & u_{43} & u_{53} & u_{63} & u_{73} \\ 0 & 0 & 0 & 0 & u_{44} & u_{54} & u_{64} & u_{74} \\ 0 & 0 & 0 & 0 & u_{45} & u_{55} & u_{65} & u_{75} \\ 0 & 0 & 0 & 0 & u_{46} & u_{56} & u_{66} & u_{76} \\ 0 & 0 & 0 & 0 & u_{47} & u_{57} & u_{67} & u_{77} \end{pmatrix}$$

  has rank at least 6. Hence the rank of the corresponding codeword of $C$ is at least 7.
- $u_{00} = 0$. It is easy to see that the rank is at least 7.

Hence $C$ is an $[\mathcal{F}, 6, 7]_q^R$ code.

*Remark 3.5* Let $\mathcal{F}_1$ bg $8 \times 9$ Ferrers diagram:

$$\mathcal{F}_1 = \begin{array}{ccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}.$$

If an $[\mathcal{F}_1, 6, 7]$-code exists, then the existence of an $[\mathcal{F}, 6, 7]$-code is a simple consequence and one doesn't even need the position (9, 9) in the matrix, where $\mathcal{F}$ is the Ferrers diagram in Example 3.4. However, it is an open question whether there exists an $[\mathcal{F}_1, 6, 7]_q^R$ code.

Let $\mathcal{F}_2$ be the following $9 \times 9$ Ferrers diagram:

$$\mathcal{F}_2 = \begin{array}{ccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet & \bullet \\ & & & & & & & \bullet & \bullet \\ & & & & & & & & \bullet \end{array}.$$

From [3, Theorem7], there exists an optimal $[\mathcal{F}_2, 6, 7]_q^R$ code for $q \geq 8$. Then the existence of an $[\mathcal{F}, 6, 7]$-code is a simple consequence and one doesn't even need the positions $(6, 5), (7, 5), (8, 5), (7, 6), (8, 6), (8, 7)$ in the matrix, where $\mathcal{F}$ is the Ferrers diagram in Example 3.4. However, this construction only works for $q \geq 8$.

Now we generalize the construction of Example 3.4.

**Theorem 3.6** *Let $1 = t_0 < t_1 < t_2 < \cdots < t_l$ be integers with $t_1 \mid t_2 \mid \cdots \mid t_l$. Let $n, \delta$ be integers for which $t_{l-1} < n - 1 \leq t_l$ and $n - t_1 + 1 < \delta \leq n - 1$. Let $\mathcal{F}$ be an $m \times n$ Ferrers diagram such that*

- $\gamma_{n-\delta} \leq t_1$;
- $\gamma_{n-\delta+1} \geq t_1$;
- $\gamma_{t_i} \geq t_{i+1}$, $i \in [1, l-1]$;
- $m = \gamma_{n-1} \geq t_l + \gamma_0$.

*Then, there exists an optimal $[\mathcal{F}, k, \delta]_q^R$ rank metric code attaining the bound from Theorem 2.4 for any $q \geq 2$.*

*Proof* Let $g_0, g_1, \ldots, g_{t_l-1}$ be a basis of $\mathbb{F}_{q^{t_l}}$ over $\mathbb{F}_q$ with $g_0 = 1$ and $g_{t_i}, \ldots, g_{t_{i+1}-1} \in \mathbb{F}_{q^{t_{i+1}}}$, $0 \leq i \leq l-1$. Then the code with generator matrix

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-2} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-2}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-2}^{[n-\delta]} \end{pmatrix}$$

is a linear $G[t_l \times (n-1), \delta - 1]_q^R$ MRD code. Note that

$$\begin{pmatrix} g_0 & g_1 & \cdots & g_{n-\delta} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-\delta}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-\delta}^{[n-\delta]} \end{pmatrix}$$

is an invertible matrix over $\mathbb{F}_{q^{t_1}}$. Then there exists a matrix $A \in GL_{n-\delta+1}(\mathbb{F}_{q^{t_1}})$ such that

$$AG = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{01} & b_{02} & \cdots & b_{0,\delta-2} \\ 0 & 1 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1,\delta-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-\delta,1} & b_{n-\delta,2} & \cdots & b_{n-\delta,\delta-2} \end{pmatrix}$$

and $b_{ij} \in \mathbb{F}_{q^{t_k}}$ for $0 \le i \le n-\delta,\ 1 \le j \le \delta-2$, where $1 \le k \le l,\ t_{k-1} < j+n-\delta+1 \le t_k$. The matrix $AG$ generates the same code as $G$.

Further, from [3, Lemma 5], there exists

$$\widehat{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{01} & b_{02} & \cdots & b_{0,\delta-2} & 0 \\ 0 & 1 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1,\delta-2} & b_{1,\delta-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-\delta,1} & b_{n-\delta,2} & \cdots & b_{n-\delta,\delta-2} & b_{n-\delta,\delta-1} \end{pmatrix}$$

such that the matrix

$$\begin{pmatrix} 1 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1,\delta-2} & b_{1,\delta-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & b_{n-\delta,1} & b_{n-\delta,2} & \cdots & b_{n-\delta,\delta-2} & b_{n-\delta,\delta-1} \end{pmatrix}$$

generates a linear $[t_l \times (n-1), \delta]_q^R$ MRD code.

Let the code $C$ be the set of all matrices of the following form:

$$C = \left\{ \left( \begin{array}{cccc} & & & \\ \multicolumn{4}{c}{\psi_{t_l}(c)} \\ \hline 0 & \ldots & 0 & u_{0,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & u_{0,s-1} \end{array} \right) \in \mathbb{F}_q^{m\times n} : c = u \cdot \widehat{G} \in \mathbb{F}_{q^{t_l}}^n,\ u = (u_0, u_1, \ldots, u_{n-\delta}) \in \mathbb{F}_{q^{t_1}}^{n-\delta+1}, \right.$$

$$\left. s = \gamma_{n-1} - t_l,\ u_{0,i} = 0 \text{ for } i \in [\gamma_0, s-1],\ \psi_{t_l}(u_i) = \begin{pmatrix} u_{i,0} \\ \vdots \\ u_{i,\gamma_i-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, 0 \le i \le n-\delta \right\}.$$

Note that $u \in \mathbb{F}_{q^{t_1}}^{n-\delta+1}$ and $b_{ij} \in \mathbb{F}_{q^{t_k}}$ for $0 \le i \le n-\delta,\ 1 \le j \le \delta-1,\ 1 \le k \le l,\ t_{k-1} < j+n-\delta+1 \le t_k$, then for $1 \le j \le \delta-2$, we have $(u \cdot \widehat{G})_{j+n-\delta} = \sum_{i=0}^{n-\delta} u_i b_{ij} \in \mathbb{F}_{q^{t_k}}$. Since $\gamma_{j+n-\delta} \ge \gamma_{t_{k-1}} \ge t_k$ for $1 \le j \le \delta-2$ and $\gamma_{n-1} \ge t_l + \gamma_0$, the code $C$ is an $[\mathcal{F}, \sum_{i=0}^{n-\delta} \gamma_i, \delta]$ rank-metric code.

On the other hand, the upper bound on the code dimension for this type of Ferrers diagrams can be obtained by deleting $\delta - 1$ rightmost columns, and therefore $\dim(\mathcal{F}, \delta) \le \sum_{i=0}^{n-\delta} \gamma_i$. Our construction attains this optimal dimension and has minimum rank distance $\delta$. □

*Remark 3.7* Taking $t_1 = 4, t_2 = 8, m = n = 9, \delta = 7$ in Theorem 3.6, we can obtain Example 3.4.

### 3.3 Construction III

Recall the definition of $\psi_m$ at the beginning of this section. For any matrix $A \in \mathbb{F}_{q^t}^{m\times n}$, define $\psi_t(A) = (\psi_t(A_{ij})) \in \mathbb{F}_q^{tm\times n}$. Then we have the following lemma.

**Lemma 3.8** *$rank_q(\psi_t(A)) \ge rank_{q^t}(A)$.*

*Proof* Let $\mathrm{rank}_{q^t}(A) = s$. If $\mathrm{rank}_q(\psi_t(A)) < s$. Then for any $s$ columns $\{A_{j_1}, A_{j_2}, \ldots, A_{j_s}\}$ of $A$, there exist $a_1, \ldots, a_s \in \mathbb{F}_q$, which are not all zero, such that $\sum_{i=1}^{s} a_i \psi_t(A_{j_i}) = 0$. Then $\psi_t(\sum_{i=1}^{s} a_i A_{j_i}) = 0$, hence $\sum_{i=1}^{s} a_i A_{j_i} = 0$, which is a contradiction. □

**Theorem 3.9** *Let $\mathcal{F}_1$ be an $m \times n$ Ferrers diagram and assume $C_1$ is an $[\mathcal{F}_1, k, \delta]_{q^t}^R$ code. Let $\mathcal{F}$ be a $tm \times n$ Ferrers diagram with $\gamma_i(\mathcal{F}) \geq t\gamma_i(\mathcal{F}_1)$ for $i \in [0, n-1]$. Then there exists an $[\mathcal{F}, tk, \geq \delta]_q^R$ code.*

*In addition, if $C_1$ is optimal, $k = \sum_{i=0}^{n-\delta} \gamma_i(\mathcal{F}_1)$ and $\gamma_i(\mathcal{F}) = t\gamma_i(\mathcal{F}_1)$ for $i \in [0, n-\delta]$, then there exists an optimal $[\mathcal{F}, tk, \delta]_q^R$ code.*

*Proof* Let $C = \{\psi_t(A) : A \in C_1\}$. Then it is easy to see that $C$ has dimension $tk$ over $\mathbb{F}_q$, and the rank distance is at least $\delta$ from Lemma 3.8. Hence $C$ is an $[\mathcal{F}, tk, \geq \delta]_q^R$ code.

For rank distance $\delta$, if $k = \sum_{i=0}^{n-\delta} \gamma_i(\mathcal{F}_1)$, then the upper bound on the code dimension for $\mathcal{F}$ can be obtained by deleting $\delta - 1$ rightmost columns. Note that $tk = \sum_{i=0}^{n-\delta} t\gamma_i(\mathcal{F}_1) = \sum_{i=0}^{n-\delta} \gamma_i(\mathcal{F})$. Hence our construction attains this optimal dimension and has minimum rank distance $\delta$. □

The following example shows a diagram in which optimal Ferrers diagram rank metric codes can be constructed by Theorem 3.9. Note that for the parameters of the following example, none of the previous constructions gives optimal codes.

*Example 3.10* Let $\mathcal{F}$ be the following $20 \times 10$ Ferrers diagram:

$$\mathcal{F} = \begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & & & & & \bullet & \bullet \\
 & & & & & & & & \bullet & \bullet \\
 & & & & & & & & & \bullet \\
 & & & & & & & & & \bullet
\end{array}.$$

For $\delta = 8$, the upper bound gives $\dim(\mathcal{F}, \delta) \leq 12$.

Let $\mathcal{F}_1$ be the following $10 \times 10$ Ferrers diagram:

$$\mathcal{F}_1 = \begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & \bullet & \bullet & \bullet \\
 & & & & & & & & \bullet & \bullet \\
 & & & & & & & & & \bullet
\end{array}.$$

Let $q \geq 3$ be a prime power, then from [3, Theorem7], there exists an optimal $[\mathcal{F}_1, 6, 8]^R_{q^2}$ code.

With Theorem 3.9, we can therefore construct an optimal $[\mathcal{F}, 12, 8]^R_q$ code for any prime power $q \geq 3$.

## 4 Conclusion

In this paper, we have presented three constructions of rank metric codes in Ferrers diagrams. The first two constructions are based on subcodes of MRD codes and their minimum rank distances are relatively large ($\delta > \frac{n}{2}$). The last construction is from known Ferrers diagram rank metric codes. Each of these constructions provides optimal codes for different types of diagrams. Our results further substantiate the Etzion–Silberstein conjecture.

Now we summarize the known results of optimal Ferrers diagram rank metric codes. Recall that $\gamma_i, i \in [0, n-1]$ denote the number of dots of diagram $\mathcal{F}$ in the $i$-th column. Then there exists an optimal $[\mathcal{F}, \delta]^R_q$ code, if the diagram $\mathcal{F}$ and minimum rank distance $\delta$ satisfy one of the following conditions:

- $\gamma_{n-\delta+1} \geq n$ ([3, Theorem 3]);
- $\gamma_{n-\delta+1} \geq n-1$ and $\gamma_{n-1} \geq n-1+\gamma_0$ ([3, Theorem 8]);
- There exist integers $t_i$, $i \in [0, l]$ such that $1 = t_0 < t_1 < t_2 < \cdots < t_l \leq m$, $t_1 \mid t_2 \mid \cdots \mid t_l$, $t_{l-1} < n \leq t_l$, $n - t_1 + 1 < \delta \leq n$, $\gamma_{n-\delta} \leq t_1$, $\gamma_{n-\delta+1} \geq t_1$ and $\gamma_{t_i} \geq t_{i+1}$, $i \in [1, l-1]$ (Theorem 3.2);
- There exist integers $t_i$, $i \in [0, l]$ such that $1 = t_0 < t_1 < t_2 < \cdots < t_l$, $t_1 \mid t_2 \mid \cdots \mid t_l$, $t_{l-1} < n-1 \leq t_l$, $n - t_1 + 1 < \delta \leq n-1$, $\gamma_{n-\delta} \leq t_1$, $\gamma_{n-\delta+1} \geq t_1$, $\gamma_{t_i} \geq t_{i+1}$, $i \in [1, l-1]$ and $m = \gamma_{n-1} \geq t_l + \gamma_0$ (Theorem 3.6);
- The diagram for codes obtained from the MDS construction by combining certain MDS codes with the same distance diagonally ([3, Theorem 7]);
- The diagram for codes obtained by combining two Ferrers diagram rank metric codes of the same dimension or the same distance appropriately ([3, Theorems 9, 10]);
- The diagram for codes obtained from optimal codes via field extension by splitting dots vertically (Theorem 3.9).

For future research, we suggest the following problems:

- The existence of optimal Ferrers diagram rank-metric codes in the $n \times n$ squares is an interesting problem. For the $n \times n$ squares, the bound of Theorem 2.4 is attained for $\delta = 2$ [4] and $\delta = 3$ ([3, Theorem 11]). Some optimal codes for $\delta = n$ were obtained in [3]. In this paper, by Theorem 3.2, we can obtain more optimal codes for $\delta = n$ (especially the case $\gamma_0 \leq k,\ \gamma_1 \geq k$ and $\gamma_k \geq n$, where $k > 1$ is a proper divisor of $n$). By Theorem 3.6, some cases for $\delta = n - 1$ can be solved (especially the case $\gamma_0 = 1,\ \gamma_1 \leq k,\ \gamma_2 \geq k$ and $\gamma_k \geq n - 1$, where $k > 1$ is a proper divisor of $n - 1$). It is interesting to get more optimal codes for these diagrams.

- The construction based on MDS codes [3, Theorem 7] only works for the field size $q$ sufficiently large. Can we give a construction which provides optimal codes for the same diagrams, but for any $q \geq 2$? This question has also been proposed in [3].
- From Remark 3.5, we found that an optimal code in a small diagram may also be an optimal code in a larger diagram. As it is a difficult problem to construct optimal Ferrers diagram rank metric codes, the next step may be to consider whether there exist optimal codes for the diagrams which are obtained by deleting some "vain" points from known diagrams (there exist optimal codes for these diagrams). For example, does there exist an $[\mathcal{F}, 6, 7]_q^R$ code for the following diagram?

$$\mathcal{F} = \begin{array}{ccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}.$$

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