Des. Codes Cryptogr. (2018) 86:803–815
https://doi.org/10.1007/s10623-017-0362-4

Combinatorial constructions of packings in Grassmannian spaces

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Received: 11 October 2016 / Revised: 9 February 2017 / Accepted: 17 April 2017 /
Published online: 26 April 2017

Abstract The problem of packing n-dimensional subspaces of m-dimensional Euclidean space such that these subspaces are as far apart as possible was introduced by Conway, Hardin and Sloane. It can be seen as a higher dimensional version of spherical codes or equiangular lines. In this paper, we first give a general construction of equiangular lines, and then present a family of equiangular lines with large size from direct product difference sets. Meanwhile, for packing higher dimensional subspaces, we give three constructions of optimal packings in Grassmannian spaces based on difference sets and Latin squares. As a consequence, we obtain many new classes of optimal Grassmannian packings.

Keywords Grassmannian packing · Equiangular line · Difference set · Latin square

Mathematics Subject Classification Primary: 52C17 · Secondary: 14M15 · 94B60

1 Introduction

Let $\mathbb{F}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. A set of N distinct lines in $\mathbb{F}^m$ through the origin, represented by vectors $x_1, \ldots, x_N$ of equal norm, is a set of equiangular lines if there is $a \in \mathbb{R}$ such that

$$|\langle x_i, x_j \rangle| = a \quad \text{for all } i \neq j.$$

The constant a is referred to the common angle between the lines.

Communicated by M. Lavrauw.

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Equiangular lines were introduced first by Haantjes [13] in 1948. Although equiangular lines have been studied for a long time, there has been a recent surge of interest since the connection with quantum information theory was established [1,24].

The number of equiangular lines in $\mathbb{C}^m$ is at most m^2 [9], and when the vectors are further constrained in $\mathbb{R}^m$ this number is at most $\frac{m(m+1)}{2}$ [19]. It is an open question, in both the complex and real cases, whether the upper bound can be attained for infinitely many m. In [17], König constructed $m^2 - m + 1$ equiangular lines in $\mathbb{C}^m$ when $m - 1$ is a prime power. de Caen [8] constructed $2(m + 1)^2/9$ equiangular lines in $\mathbb{R}^m$, where $(m + 1)/3$ is twice a power of 4. Recently, Jedwab et al. [14,15] and Greaves et al. [12] gave some constructions of large sets of equiangular lines from sets of mutually unbiased bases.

Next to lines, packings of n-dimensional subspaces have also been investigated [6,10,23]. The goal is to find a set of n-dimensional subspaces $U_1, \ldots, U_N$ of $\mathbb{F}^m$, such that they are as far apart as possible. These packing problems have applications in coding theory [2] and quantum information theory [29].

The Grassmannian space $G_{\mathbb{F}}(m, n)$ is the set of all n-dimensional subspaces of $\mathbb{F}^m$. For an n-dimensional subspace U of $\mathbb{F}^m$, let A be an $n \times m$ generator matrix for U, whose rows are orthonormal vectors spanning U. Then the projection from $\mathbb{F}^m$ to U can be represented by the matrix $P_U = A^*A$, where A^* denotes the adjoint of A or the transpose of A in the real case. For two n-dimensional subspaces U, V of $\mathbb{F}^m$, the chordal distance is defined by

$$d^2(U, V) = n - \mathrm{tr}(P_U P_V).$$

The representation of a subspace by a projection matrix does not depend on the choice of a basis of a subspace, and we have an inclusion of the Grassmannian space into the space of symmetric or Hermitian $m \times m$ matrices equipped with the Hilbert-Schmidt norm. It turns out that matrices corresponding to projections lie on a sphere of radius $\frac{\sqrt{m}}{2}$ centered at $\frac{1}{2}I$. Therefore, every Grassmannian packing with chordal distance corresponds to a spherical code. Then the simplex and orthoplex Rankin bounds of spherical codes can be translated to the Grassmannian packings. Denote

$$d_{\mathbb{F}}(m) = \begin{cases} \frac{(m+2)(m-1)}{2}, & \text{if } \mathbb{F} = \mathbb{R}; \\ m^2 - 1, & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Then for $N \leq d_{\mathbb{F}}(m) + 1$, we have the simplex bound:

$$\min_{i \neq j} d^2(U_i, U_j) \leq \frac{n(m-n)}{m} \frac{N}{N-1},$$

where the equality holds for a simplex configuration, i.e., the distance between any pair of subspaces is equal to a constant.

For $N > d_{\mathbb{F}}(m) + 1$, we have the orthoplex bound:

$$\min_{i \neq j} d^2(U_i, U_j) \leq \frac{n(m-n)}{m}.$$

If the equality is achieved then $N \leq 2d_{\mathbb{F}}(m)$. We summarize the above discussion as follows.

Definition 1 Let $\mathfrak{U}$ be a set of n-dimensional subspaces $U_1, \ldots, U_N$ of $\mathbb{F}^m$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). Then $\mathfrak{U}$ is called a *simplex packing* of $G_{\mathbb{F}}(m, n)$ if $N \leq d_{\mathbb{F}}(m) + 1$ and $\min_{i \neq j} d^2(U_i, U_j) = \frac{n(m-n)}{m} \frac{N}{N-1}$. $\mathfrak{U}$ is called an *orthoplex packing* of $G_{\mathbb{F}}(m, n)$ if $N > d_{\mathbb{F}}(m) + 1$ and $\min_{i \neq j} d^2(U_i, U_j) = \frac{n(m-n)}{m}$. Both simplex packings and orthoplex packings are called *optimal packings*.

Although Grassmannian packings have been studied for 20 years, there are only a few optimal packings. Several optimal examples were provided in [6]. In [4,25], the authors gave some constructions of optimal packings in Grassmannian spaces based on the Clifford group and Barnes–Wall lattices. In [16], Kocák and Niepel proposed several families of optimal packings in Grassmannian spaces by using Hadamard matrices. Recently, Bodmann and Haas [3] presented a construction of orthoplex packings in Grassmannian spaces from mutually unbiased bases and block designs.

In [17], the author used difference sets to construct equiangular lines, and the results in [14,15] can be obtained from relative difference sets. The main technique from difference sets is the fact that $|\{|\chi(D)| : \chi \in \widehat{G}\backslash\{\chi_0\}\}| = 1$ (or 2) if D is a difference set (or relative difference set, respectively) in a group G. Note that $|\chi(D)|$ $(\chi \in \widehat{G}\backslash\{\chi_0\})$ takes three values if D is a direct product difference set. In this paper, we extend this idea and present a class of $O(d^2)$ equiangular lines in $\mathbb{C}^d$ from direct product difference sets. Later, we consider packing higher dimensional subspaces. We present two constructions of simplex packings of Grassmannian spaces from difference sets. Since there are a lot of constructions of difference sets, we obtain many new families of optimal packings in Grassmannian spaces. In particular, we give a new explanation of the construction in [4]. Meanwhile, we construct a class of optimal packings from Latin squares.

This paper is organized as follows. In Sect. 2, we recall some basics about difference sets. Section 3 gives a construction of equiangular lines. In Sect. 4, we give three constructions of simplex Grassmannian packings. Section 5 concludes the paper.

2 Preliminaries

Let G be a finite group. The group ring $\mathbb{Z}[G]$ is a free abelian group with a basis $\{g \mid g \in G\}$. For any set A whose elements belong to G (A may be a multiset), we identify A with the group ring element $\sum_{g\in G} d_g g$, where d_g is the multiplicity of g appearing in A. Given any $A = \sum_{g\in G} d_g g \in \mathbb{Z}[G]$, we define $A^{(-1)} = \sum_{g\in G} d_g g^{-1}$, in which g^{-1} is the inverse of g with respect to the operation of group G. Addition and multiplication in group rings are defined as:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

and

$$\sum_{g\in G} a_g g \sum_{g\in G} b_g g = \sum_{g\in G} \left(\sum_{h\in G} a_h b_{h^{-1}g} \right) g.$$

For a finite abelian group G, denote its character group by $\widehat{G}$. The identity element of $\widehat{G}$ is called the principal character, denoted by χ_0. For any $A = \sum_{g\in G} d_g g$ and $\chi \in \widehat{G}$, define $\chi(A) = \sum_{g\in G} d_g \chi(g)$. The following properties of characters are well known.

Theorem 2 (Orthogonality relations) *Let G be an abelian group of order v with identity e. Then*

$$\sum_{\chi\in\widehat{G}} \chi(g) = \begin{cases} 0, & \text{if } g \neq e; \\ v, & \text{if } g = e, \end{cases}$$

and

$$\sum_{g\in G}\chi(g)=\begin{cases}0, & \text{if } \chi\neq\chi_0;\\ v, & \text{if } \chi=\chi_0.\end{cases}$$

A subset D of size k in a group $(G,\cdot)$ with order v is called a (v,k,λ) difference set in $(G,\cdot)$ if the list of products $d_1\cdot d_2^{-1}$ with $d_1,d_2\in D$ contains every non-identity element of G exactly λ times. In the language of group rings, we have

$$DD^{(-1)}=(k-\lambda)1_G+\lambda G,$$

where 1_G is the identity element of G.

By definition, if D is a (v,k,λ) difference set in $(G,\cdot)$, then

$$|D\cap(D\cdot x)|=\lambda$$

for every non-identity element $x\in G$, and

$$k(k-1)=\lambda(v-1).$$

A difference set D in an abelian group $(G,+)$ is called a skew Hadamard difference set if it has parameters $(q,\frac{q-1}{2},\frac{q-3}{4})$ and G is the union of D, $-D$ and $\{0_G\}$, where 0_G is the identity element of abelian group G. It can be shown that a skew Hadamard difference set exists if and only if $q\equiv 3 \pmod 4$ is a prime power.

Let $G=A\times B$ be the direct product of two groups A and B with $|A|=a$ and $|B|=b$ $(a,b\geq 2)$. Let $D\subseteq G$ with size k such that every element of $G\backslash\{A\cup B\}$ can be represented in exactly λ (≥ 1) ways in the form $d_1^{-1}d_2$, where $d_1,d_2\in D$. Furthermore, suppose that no nonidentity element of $A\cup B$ can be so represented. Then we call D an (a,b,k,λ) direct product difference set in G. Using the language of group rings, an (a,b,k,λ) direct product difference set in G can be expressed as:

$$DD^{(-1)}=(k+\lambda)1_G+\lambda(G-A-B).$$

Direct product difference sets can be viewed as a generalization of difference sets. We refer to [22, Sect. 5.3] for more information on these combinatorial objects.

3 A construction of equiangular lines

In this section, we give a construction of equiangular lines from difference sets. We first give a general construction.

Theorem 3 *Let $B_i=(b_{i1},\ldots,b_{im})$ $(1\leq i\leq s)$ be $n\times m$ matrices, where $b_{ij}\in\mathbb{F}^n$ $(\mathbb{F}=\mathbb{C}$ or $\mathbb{R})$. Suppose there exist $a,d,e,f\in\mathbb{R}$ and $b,c\in\mathbb{F}$ such that the following conditions hold:*

1. *For any $1\leq i\leq s,\ 1\leq j\leq m$, $\langle b_{ij},b_{ij}\rangle=a$.*
2. *For any $1\leq i\leq s,\ 1\leq j\neq k\leq m$, $\langle b_{ij},b_{ik}\rangle=b$.*
3. *For any $1\leq i\neq k\leq s,\ 1\leq j\leq m$, $\langle b_{ij},b_{kj}\rangle=c$.*
4. *For any $1\leq i\neq k\leq s,\ 1\leq j\neq l\leq m$, $|\langle b_{ij},b_{kl}\rangle|=d$.*
5. *$|b+f^2|=|c+e^2|=d$.*

Then there exist ms equiangular lines in $\mathbb{F}^{n+m+s}$.

Proof We build the following matrix whose columns will form the desired configuration of equiangular lines.

$$M = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ eI_m & eI_m & \cdots & eI_m \\ fJ_m & \overrightarrow{0} & \cdots & \overrightarrow{0} \\ \overrightarrow{0} & fJ_m & \cdots & \overrightarrow{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overrightarrow{0} & \overrightarrow{0} & \cdots & fJ_m \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix, $\overrightarrow{0} = (\underbrace{0, 0, \ldots, 0}_{m})$ and $J_m = (\underbrace{1, 1, \ldots, 1}_{m})$. □

Now we are going to give a construction of matrices satisfying the conditions of Theorem 3. Let $G = \mathbb{F}_q^+ \times \mathbb{F}_q^*$, then the set $D = \{(x, x) \in \mathbb{F}_q^+ \times \mathbb{F}_q^* | x \in \mathbb{F}_q^*\}$ is a $(q, q-1, q-1, 1)$ direct product difference set in G relative to $N_1 = \{(0, x) | x \in \mathbb{F}_q^*\}$ and $N_2 = \{(x, 1) | x \in \mathbb{F}_q^+\}$ [11]. In the language of group rings, we have

$$DD^{(-1)} = q \cdot 1_G + G - N_1 - N_2,$$

where $D^{(-1)}$ denotes the group ring element identifying the set $\{(-x, y^{-1}) : (x, y) \in D\}$.

If χ is a character of an abelian group, by Theorem 2, we have

$$\chi(DD^{(-1)}) = \begin{cases} (q-1)^2, & \text{if } \chi = \chi_0; \\ 1, & \text{if } \chi|_{N_1} = \chi_0 \text{ and } \chi \neq \chi_0; \\ 0, & \text{if } \chi|_{N_2} = \chi_0 \text{ and } \chi \neq \chi_0; \\ q, & \text{if } \chi|_{N_1} \neq \chi_0 \text{ and } \chi|_{N_2} \neq \chi_0. \end{cases}$$

Moreover, we can compute to obtain that

$$\chi(D) = \begin{cases} -1, & \text{if } \chi|_{N_1} = \chi_0 \text{ and } \chi \neq \chi_0; \\ 0, & \text{if } \chi|_{N_2} = \chi_0 \text{ and } \chi \neq \chi_0. \end{cases}$$

Theorem 4 *Let q be a prime power and $s \leq q$ be a positive integer. Then there exist $(q-1)s$ equiangular lines in $\mathbb{C}^{2q-2+s}$.*

Proof Following the notations above, it is easy to see that $\widehat{G} \cong \widehat{\mathbb{F}_q^+} \times \widehat{\mathbb{F}_q^*}$. Let $\widehat{\mathbb{F}_q^+} = \{\psi_1, \ldots, \psi_q\}$, $\widehat{\mathbb{F}_q^*} = \{\chi_1, \ldots, \chi_{q-1}\}$ and denote $D = \{d_1, d_2, \ldots, d_{q-1}\}$. Let $B_i = (\psi_i \chi_k(d_j))_{\substack{1 \leq j \leq q-1 \\ 1 \leq k \leq q-1}}$ for $1 \leq i \leq s$, then $b_{ij} = (\psi_i \chi_j(d_1), \ldots, \psi_i \chi_j(d_{q-1}))^\top$, where $A^\top$ denotes the transpose of A. We have $\langle b_{ij}, b_{rt} \rangle = \psi_i \psi_r^{-1} \chi_j \chi_t^{-1}(D)$, then the matrices B_i satisfy the conditions of Theorem 3 with $a = q-1,\ b = -1,\ c = 0,\ d = \sqrt{q}$. Hence the result follows from Theorem 3 by taking $f = \sqrt{1 + \sqrt{q}}$ and $e = \sqrt[4]{q}$. □

In particular, we have the following corollary.

Corollary 5 *There exist $q^2 - q$ equiangular lines in $\mathbb{C}^{3q-2}$, where q is a prime power.*

4 Three constructions of simplex Grassmannian packings

4.1 The first construction

Theorem 6 *If there exists a difference set D in a group $(G,\cdot)$ with parameters (v,k,λ), then there exists an optimal simplex packing of v real subspaces of dimension k in the real Grassmannian space $G_{\mathbb{R}}(v,k)$.*

Proof Let e_g $(g\in G)$ be the standard orthonormal basis of $\mathbb{R}^v$. Let U_1 be the k-dimensional subspace spanned by the vectors

$$e_d,\ d\in D.$$

Then we can obtain $v-1$ further subspaces U_h by permutations $e_g\mapsto e_{g\cdot h}$ $(h\in G)$.

Let $D=\{d_1,\ldots,d_k\}$, P_h be the projection matrix of U_h and $E_h=(e_{d_1\cdot h},\ldots,e_{d_k\cdot h})$. Then E_h^* is the generator matrix for U_h, whose rows are orthonormal vectors spanning U_h. Hence $P_h=E_hE_h^*$.

For two distinct subspaces U_g and U_h, we can compute to obtain that

$$\begin{aligned} d^2(U_g,U_h) &= k-\mathrm{tr}(P_gP_h)\\ &= k-\mathrm{tr}(E_gE_g^*E_hE_h^*)\\ &= k-\mathrm{tr}((E_g^*E_hE_h^*)E_g)\\ &= k-\mathrm{tr}((E_g^*E_h)(E_g^*E_h)^*)\\ &= k-|(D\cdot g)\cap(D\cdot h)|\\ &= k-\lambda. \end{aligned}$$

Note that $k(k-1)=\lambda(v-1)$, then the simplex bound on minimal distance for v k-subspaces in $\mathbb{R}^v$ is

$$\min_{U\neq V} d^2(U,V)\le \frac{k(v-k)}{v}\,\frac{v}{v-1}=k-\lambda.$$

□

We give an example to illustrate our construction.

Example 7 Let $v=7$ and $D=\{1,2,4\}$. Then D is a $(7,3,1)$ difference set in $(\mathbb{Z}_7,+)$. The subspace U_0 is generated by

$$\begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0 \end{pmatrix},$$

and the other subspaces are generated by cycling the seven coordinates. It can be checked that each pair of subspaces has the same distance.

Now we give several new classes of optimal packings from known difference sets.

Example 8 (Skew Hadamard difference sets [21]) The skew Hadamard difference set is a difference set with parameters $(q,\frac{q-1}{2},\frac{q-3}{4})$ and G being the union of D, $-D$ and 1_G, where $q\equiv 3 \pmod 4$ is a prime power. Then from Theorem 6, there exists an optimal simplex packing of q real subspaces of dimension $\frac{q-1}{2}$ in the real Grassmannian space $G_{\mathbb{R}}(q,\frac{q-1}{2})$ for every prime power $q\equiv 3 \pmod 4$.

Example 9 (Difference sets with Singer parameters [26]) A difference set is said to have Singer parameters if its parameters are of the form $(\frac{q^m-1}{q-1}, \frac{q^{m-1}-1}{q-1}, \frac{q^{m-2}-1}{q-1})$, where q is a prime power and m is a positive integer. There are a lot of constructions of difference sets with Singer parameters. Then from Theorem 6, there exists an optimal simplex packing of $\frac{q^m-1}{q-1}$ real subspaces of dimension $\frac{q^{m-1}-1}{q-1}$ in the real Grassmannian space $G_{\mathbb{R}}(\frac{q^m-1}{q-1}, \frac{q^{m-1}-1}{q-1})$ for prime power q and positive integer m.

Example 10 (McFarland difference sets [20]) The McFarland difference set is a difference set with parameters $(q^{m+1}(1+\frac{q^{m+1}-1}{q-1}), q^m\frac{q^{m+1}-1}{q-1}, q^m\frac{q^m-1}{q-1})$, where q is a prime power and m is a positive integer. Then from Theorem 6, there exists an optimal simplex packing of $q^{m+1}(1+\frac{q^{m+1}-1}{q-1})$ real subspaces of dimension $q^m\frac{q^{m+1}-1}{q-1}$ in the real Grassmannian space $G_{\mathbb{R}}(q^{m+1}(1+\frac{q^{m+1}-1}{q-1}), q^m\frac{q^{m+1}-1}{q-1})$ for prime power q and positive integer m.

We close this subsection with some remarks.

Remark 11
1. Since there are a lot of difference sets, for example: Menon difference sets [7], cyclotomic difference sets [18], twin-prime difference sets [28] and so on, we can obtain more optimal packings in Grassmannian spaces from Theorem 6.
2. There are a lot of inequivalent skew Hadamard difference sets [27], hence there are a lot of choices for the construction above with the same parameters. This is similar for other types of difference sets.

4.2 The second construction

In this subsection, we give our second construction of optimal packings. As a preparation, we need the following definition.

Definition 12 An $N \times N$ matrix H is a complex Hadamard matrix if $|H_{jk}| = 1$ for $j, k = 1, 2, \ldots, N$ and $HH^* = NI_N$, where I_N is the identity matrix.

Complex Hadamard matrices exist for any natural number N. For instance the matrix $H = (e^{2\pi i(j-1)(k-1)/N})_{j,k=0,\ldots,N-1}$ is an $N \times N$ complex Hadamard matrix.

Theorem 13 *There exists an optimal simplex packing of* $\frac{q(q+1)}{2}$ *complex subspaces of dimension* $\frac{q-1}{2}$ *in the complex Grassmannian space* $G_{\mathbb{C}}(q, \frac{q-1}{2})$, *where* $q \equiv 3 \pmod 4$ *is a prime power.*

Proof Let $q \equiv 3 \pmod 4$ be a prime power, $k = \frac{q-1}{2}$, $(G, +)$ be an abelian group with order q and $D = \{d_1, d_2, \ldots, d_{\frac{q-1}{2}}\}$ be a skew Hadamard difference set in $(G, +)$ with parameters $(q, \frac{q-1}{2}, \frac{q-3}{4})$. Then $-D$ is also a difference set and $D \cup (-D) \cup \{0_G\} = G$. For any non-identity element $x \in G$, we have

$$\begin{aligned}
&|D \cap (-D + x)| + |(D + x) \cap -D| \\
&\quad = |(D \cup -D) \cap ((D \cup -D) + x)| - |D \cap (D + x)| - |-D \cap (-D + x)| \\
&\quad = q - 2 - \frac{q-3}{4} - \frac{q-3}{4} \\
&\quad = \frac{q-1}{2}.
\end{aligned}$$

Let $C=\frac{1+\sqrt{q+2}}{\sqrt{q+1}}$ and e_g ($g\in G$) be the standard orthonormal basis of $\mathbb{C}^q$. Let $H=(H_{ij})_{0\le i,j\le\frac{q-1}{2}}$ be a complex Hadamard matrix, we assume that $H_{i0}=H_{0j}=1$ for all i, j. Let U_{t0} ($0\le t\le\frac{q-1}{2}$) be the $\frac{q-1}{2}$-dimensional subspaces spanned by the vectors

$$e_{d_i}+\overline{H_{it}}Ce_{-d_i},\ d_i\in D.$$

For each U_{t0}, we obtain $q-1$ further subspaces U_{th} by permutations $e_g\mapsto e_{g+h}$ ($h\in G$).

Let P_{th} be the projection matrix of U_{th} and $A_{th}=(\frac{e_{d_1+h}+H_{1t}Ce_{-d_1+h}}{\sqrt{1+C^2}},\ldots,\frac{e_{d_k+h}+H_{kt}Ce_{-d_k+h}}{\sqrt{1+C^2}})$. Then A^*_{th} is the generator matrix for U_{th}, whose rows are orthonormal vectors spanning U_{th}. Hence $P_{th}=A_{th}A^*_{th}$.

Since $D\cap(-D)=\emptyset$, we have $A^*_{th}A_{sh}=\mathrm{diag}(\frac{1+C^2\overline{H_{1t}}H_{1s}}{1+C^2},\ldots,\frac{1+C^2\overline{H_{kt}}H_{ks}}{1+C^2})$ for $t\ne s$. Then for two subspaces U_{tg} and U_{sg} ($t\ne s$), we have

$$\begin{aligned}
d^2(U_{tg},U_{sg})&=\frac{q-1}{2}-\mathrm{tr}(P_{tg}P_{sg})\\
&=\frac{q-1}{2}-\mathrm{tr}(A_{tg}A^*_{tg}A_{sg}A^*_{sg})\\
&=\frac{q-1}{2}-\mathrm{tr}((A^*_{tg}A_{sg}A^*_{sg})A_{tg})\\
&=\frac{q-1}{2}-\mathrm{tr}((A^*_{tg}A_{sg})(A^*_{tg}A_{sg})^*)\\
&=\frac{q-1}{2}-\frac{1}{(1+C^2)^2}\sum_{i=1}^{k}(1+C^2\overline{H_{it}}H_{is})\big(1+C^2H_{it}\overline{H_{is}}\big)\\
&=\frac{q-1}{2}-\frac{1}{(1+C^2)^2}\sum_{i=1}^{k}(1+C^2\overline{H_{it}}H_{is}+C^2H_{it}\overline{H_{is}}+C^4\big)\\
&=\frac{q-1}{2}-\frac{q-1}{2}\frac{1+C^4}{(1+C^2)^2}+\frac{2C^2}{(1+C^2)^2}\\
&=\frac{(q+1)^2}{4(q+2)}.
\end{aligned}$$

Let $E_{th1}=(\frac{e_{d_1+h}}{\sqrt{1+C^2}},\ldots,\frac{e_{d_k+h}}{\sqrt{1+C^2}})$ and $E_{th2}=(\frac{H_{1t}Ce_{-d_1+h}}{\sqrt{1+C^2}},\ldots,\frac{H_{kt}Ce_{-d_k+h}}{\sqrt{1+C^2}})$, then $A_{th}=E_{th1}+E_{th2}$. Since $D\cap(-D)=\emptyset$, we have $E^*_{th1}E_{th2}=0$. For $g\ne h$, $A^*_{tg}A_{sh}=E^*_{tg1}E_{sh1}+E^*_{tg1}E_{sh2}+E^*_{tg2}E_{sh1}+E^*_{tg2}E_{sh2}$. Then

$$\begin{aligned}
\mathrm{tr}((A^*_{tg}A_{sh})(A^*_{tg}A_{sh})^*)&=\mathrm{tr}((E^*_{tg1}E_{sh1})(E^*_{tg1}E_{sh1})^*+(E^*_{tg1}E_{sh2})(E^*_{tg1}E_{sh2})^*\\
&\quad+(E^*_{tg2}E_{sh1})(E^*_{tg2}E_{sh1})^*+(E^*_{tg2}E_{sh2})(E^*_{tg2}E_{sh2})^*).
\end{aligned}$$

For two subspaces U_{tg} and U_{sh} ($g\ne h$), using the properties of skew Hadamard difference sets, we get

$$\begin{aligned}
d^2(U_{tg},U_{sh})&=\frac{q-1}{2}-\mathrm{tr}(P_{tg}P_{sh})\\
&=\frac{q-1}{2}-\mathrm{tr}(A_{tg}A^*_{tg}A_{sh}A^*_{sh})\\
&=\frac{q-1}{2}-\mathrm{tr}((A^*_{tg}A_{sh}A^*_{sh})A_{tg})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q-1}{2} - \operatorname{tr}((A_{tg}^* A_{sh})(A_{tg}^* A_{sh})^*) \\
&= \frac{q-1}{2} - \operatorname{tr}\Big((E_{tg1}^* E_{sh1})(E_{tg1}^* E_{sh1})^* + (E_{tg1}^* E_{sh2})(E_{tg1}^* E_{sh2})^* \\
&\quad + (E_{tg2}^* E_{sh1})(E_{tg2}^* E_{sh1})^* + (E_{tg2}^* E_{sh2})(E_{tg2}^* E_{sh2})^*\Big) \\
&= \frac{q-1}{2} - |(D+g)\cap(D+h)|\frac{1+C^4}{(1+C^2)^2} - \Big(|(D+g)\cap(-D+h)| \\
&\quad + |(D+h)\cap(-D+g)|\Big)\frac{C^2}{(1+C^2)^2} \\
&= \frac{q-1}{2} - \frac{q-3}{4}\frac{1+C^4}{(1+C^2)^2} - \frac{q-1}{2}\frac{C^2}{(1+C^2)^2} \\
&= \frac{(q+1)^2}{4(q+2)}.
\end{aligned}
$$

We can also compute the simplex bound on minimal distance for $\frac{q(q+1)}{2}$ $\frac{q-1}{2}$-subspaces in $\mathbb{C}^q$:

$$
\min_{U\neq V} d^2(U,V) \leq \frac{\frac{q-1}{2}\left(q - \frac{q-1}{2}\right)}{q}\,\frac{\frac{q(q+1)}{2}}{\frac{q(q+1)}{2}-1} = \frac{(q+1)^2}{4(q+2)}.
$$

□

Remark 14 1. A similar result of Theorem 13 has appeared in [4,16]. In this subsection, we give a new construction of such packings by skew Hadamard difference sets.

2. In fact, suppose we have two difference sets D_1, D_2 in abelian group $(G, +)$ with the same parameters (v, k, λ) satisfying the following conditions:

   (a) $D_1 \cap D_2 = \emptyset$;
   (b) for any non-identity $x \in G$, $|D_1 \cap (D_2 + x)| + |(D_1 + x) \cap D_2| = \mu$ for some constant μ;
   (c) $D_2 = \sigma(D_1)$ for some $\sigma \in \mathrm{Aut}(G)$.

   Then the above construction also works by changing D and $-D$ to D_1 and D_2, respectively. Unfortunately, if difference sets D_1, D_2 satisfy these conditions, then D_1, D_2 must be difference sets with skew Hadamard parameters. This can be seen as follows.
   If D_1, D_2 are two difference sets satisfying the conditions above, then $|(D_1 \cup D_2) \cap ((D_1 \cup D_2) + x)| = \mu + 2\lambda$ for any non-identity $x \in G$. Therefore $D_1 \cup D_2$ is also a difference set. From the elementary identity of difference sets, we have

$$
\begin{aligned}
k(k-1) &= \lambda(v-1), \\
2k(2k-1) &= (\mu + 2\lambda)(v-1).
\end{aligned}
$$

   Thus $(v-1)|\gcd(k(k-1), 2k(2k-1))$, and then $v - 1 \leq 2k \leq v$. Therefore $k = \frac{v-1}{2}$, and D_1, D_2 are difference sets with parameters $(v, \frac{v-1}{2}, \frac{v-3}{4})$.

4.3 The third construction

In this subsection, we give a construction of optimal packings from Latin squares.

Definition 15 A Latin square L of side n on the symbol set $\{0, 1, \ldots, n-1\}$ is an $n \times n$ array such that each symbol occurs exactly once in each row and exactly once in each column. A Latin square L of side n is symmetric if $L(i, j) = L(j, i)$ for all $0 \le i, j \le n-1$.

There have been many research works concerning Latin squares. For a survey of recent progress in this area we refer the reader to [5]. Now we state our main construction.

Theorem 16 *Assume that there exists a symmetric Latin square L of side st on the symbol set* $\{0, 1, \ldots, st-1\}$ *satisfying the following conditions:*

(1) For $m = 0, s, 2s, \ldots, s(t-1)$*, the number of solutions to* $L(x, x) = m$ *is* s*;*
(2) For $m = 0, s, 2s, \ldots, s(t-1)$*,* $i = 0, 1, \ldots, st-1$*, let* $u(i)$ *be the unique number such that* $L(i, u(i)) = m$*. We have* $\{(i - u(i)) \pmod{st} : i = 0, 1, \ldots, st-1\} = \{0, a, \ldots, st-a\}$ *for some* $a|st$*,* $a \le s$ *and each* ja $(j = 0, 1, \ldots, \frac{st}{a} - 1)$ *occurs exactly* a *times.*

Then there exists an optimal simplex packing of t^2 *complex subspaces of dimension* $\frac{s(t+1)}{2}$ *in the complex Grassmannian space* $G_{\mathbb{C}}(st, \frac{s(t+1)}{2})$.

Proof From the property of symmetric Latin square and the first condition, we have $2|s(t+1)$.

Let ω be a primitive st-th root of unity. For $m = 0, s, 2s, \ldots, s(t-1)$ and $l = 0, 1, \ldots, t-1$, we define matrices

$$(R_{m,l})_{i,j} = \begin{cases} \omega^{(i-j)l}, & \text{if } L(i, j) = m; \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \le i, j \le st-1$.

It is easy to check that $R_{m,l}$ satisfies the following conditions:

(1) $R^*_{m,l} = R_{m,l}$;
(2) $\operatorname{tr}(R_{m,l}) = s$;
(3) $R^2_{m,l} = I$.

Next we compute $\operatorname{tr}(R_{m,l}R_{m',l'})$ for distinct pairs $(m, l) \ne (m', l')$. If $m \ne m'$, then $\operatorname{tr}(R_{m,l}R_{m',l'}) = 0$ since the nonzero entries of $R_{m,l}$ and $R_{m',l'}$ do not overlap. If $m = m'$, from the second condition, we have

$$\begin{aligned} \operatorname{tr}(R_{m,l}R_{m',l'}) &= \sum_{i=0}^{st-1} \omega^{(i-u(i))(l-l')} \\ &= a \sum_{i=0}^{\frac{st}{a}-1} \omega^{ai(l-l')} = 0. \end{aligned}$$

Let $P_{m,l} = \frac{1}{2}(R_{m,l} + I)$, then from the properties of $R_{m,l}$, we have

(1) $P^*_{m,l} = P_{m,l}$;
(2) $\operatorname{tr}(P_{m,l}) = \frac{s(t+1)}{2}$;
(3) $P^2_{m,l} = P_{m,l}$;
(4) for $(m, l) \ne (m', l')$, we have $\operatorname{tr}(P_{m,l}P_{m',l'}) = \frac{s}{2} + \frac{st}{4}$.

Let $U_{m,l}$ be the subspace corresponding to the projection matrix $P_{m,l}$, then the distance between two subspaces is

$$d^2(U_{m,l}, U_{m',l'}) = \frac{s(t+1)}{2} - \operatorname{tr}(P_{m,l}P_{m',l'}) = \frac{st}{4}.$$

We can also compute the simplex bound on minimal distance for $t^2\ \frac{s(t+1)}{2}$-subspaces in $\mathbb{C}^{st}$:

$$\min_{U \neq V} d^2(U, V) \leq \frac{\frac{s(t+1)}{2}\left(st - \frac{s(t+1)}{2}\right)}{st} \frac{t^2}{t^2 - 1} = \frac{st}{4}.$$

Therefore, the t^2 matrices $P_{m,l}$ ($m = 0, s, 2s, \dots, s(t-1)$, $l = 0, 1, \dots, t-1$) form an optimal simplex packing in the complex Grassmannian space $G_{\mathbb{C}}(st, \frac{s(t+1)}{2})$. □

Remark 17 Let $s = 1$, t be an odd integer, $L(i, j) = (i + j) \pmod t$. By Theorem 16, there exists an optimal simplex packing of t^2 complex subspaces of dimension $\frac{t+1}{2}$ in the complex Grassmannian space $G_{\mathbb{C}}(t, \frac{t+1}{2})$. This result was obtained in [16].

Now we give a construction of symmetric Latin squares satisfying the conditions of Theorem 16. Let $G(n) = \{0, 1, \dots, n-1\}$, s be a positive even integer and t be a positive odd integer. Define $f_s(x, y) : G(s) \times G(s) \to G(s)$ by

$$f_s(x, y) = \begin{cases} 0, & \text{if } x = y; \\ a + 1, & \text{if } a \in G(s-1),\ \{x, y\} = \{a, s-1\}; \\ a + 1, & \text{if } a, x, y \in G(s-1),\ x \neq y,\ x + y \equiv 2a \pmod{s-1}. \end{cases}$$

Let $g_t(x, y) = (x + y) \pmod t$ be a map from $G(t) \times G(t)$ to $G(t)$. Then we define $L_{s,t}(x, y) : G(st) \times G(st) \to G(st)$ by

$$L_{s,t}(sx_1 + x_2, sy_1 + y_2) = sg_t(x_1, y_1) + f_s(x_2, y_2) \text{ for } x_1, y_1 \in G(t), x_2, y_2 \in G(s).$$

It can be verified that $L_{s,t}(x, y)$ satisfies the following conditions:

(1) $L_{s,t}(x, y) = L_{s,t}(y, x)$;
(2) for $m = 0, s, 2s, \dots, s(t-1)$, the number of solutions to $L_{s,t}(x, x) = m$ is s;
(3) for any $i, k \in G(st)$, there exists a unique $j \in G(st)$ such that $L_{s,t}(i, j) = k$.

Then for $m = 0, s, 2s, \dots, s(t-1)$, $i \in G(t)$ and $j \in G(s)$, there exist unique $u(i) \in G(t)$, $u(j) \in G(s)$ such that $L_{s,t}(si + j, su(i) + u(j)) = m$. From the definition of $L_{s,t}(x, y)$, we have $u(i) = (\frac{m}{s} - i) \pmod t$ and $u(j) = j$. Hence $\{(si + j - su(i) - u(j)) \pmod{st} : i \in G(t), j \in G(s)\} = \{0, s, 2s, \dots, s(t-1)\}$ and each ks ($k = 0, 1, \dots, t-1$) occurs exactly s times. Therefore, $L_{s,t}(x, y)$ forms a symmetric Latin square satisfying the conditions of Theorem 16. Then we have the following corollary.

Corollary 18 *There exists an optimal simplex packing of t^2 complex subspaces of dimension $\frac{s(t+1)}{2}$ in the complex Grassmannian space $G_{\mathbb{C}}(st, \frac{s(t+1)}{2})$, where s is a positive even integer and t is a positive odd integer.*

5 Conclusion

In [17], König constructed $d^2 - d + 1$ equiangular lines in $\mathbb{C}^d$ by difference sets, where $d - 1$ is a prime power. Recently, Jedwab et al. [14,15] gave some constructions of large sets of equiangular lines from sets of mutually unbiased bases. Their results can also be obtained from relative difference sets through a similar approach as that of Sect. 3. The main technique from difference sets is the fact that $|\{|\chi(D)| : \chi \in \widehat{G}\backslash\{\chi_0\}\}| = 1$ (or 2) if D is a difference set (or relative difference set, respectively) in group G. Note that $|\chi(D)|$ ($\chi \in \widehat{G}\backslash\{\chi_0\}$) takes three values if D is a direct product difference set. In this paper, we construct a family of $O(d^2)$ equiangular lines in $\mathbb{C}^d$. This method may be generalized if we can find a subset D in an abelian group G with $|\chi(D)|$ ($\chi \in \widehat{G}\backslash\{\chi_0\}$) taking few values.

In Sect. 4, we give three constructions of simplex Grassmannian packings. Based on the first construction, any difference set will automatically give a simplex Grassmannian packing. The second construction by skew Hadamard difference sets gives a new explanation of the construction in [4]. The third construction presents many new optimal Grassmannian packings from certain Latin squares.

It seems that combinatorial configurations are good sources to construct optimal Grassmannian packings. We expect that more optimal packings can be produced via classical combinatorial configurations.

Acknowledgements The authors express their gratitude to the anonymous reviewers for their detailed and constructive comments which are very helpful to the improvement of the presentation of this paper. Research supported by the National Natural Science Foundation of China under Grant Nos. 11431003 and 61571310, Beijing Hundreds of Leading Talents Training Project of Science and Technology, and Beijing Municipal Natural Science Foundation.

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