Perfect and Quasi-Perfect Codes Under the l_p Metric

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***Abstract*— A long-standing conjecture of Golomb and Welch, raised in 1970, states that there is no perfect r error correcting Lee code of length n for $n \geq 3$ and $r > 1$. In this paper, we study perfect codes in $\mathbb{Z}^n$ under the l_p metric, where $1 \leq p < \infty$. We show some nonexistence results of linear perfect l_p codes for $p = 1$ and $2 \leq p < \infty$, $r = 2^{1/p}, 3^{1/p}$. We also give an algebraic construction of quasi-perfect l_p codes for $p = 1, r = 2$, and $2 \leq p < \infty, r = 2^{1/p}$.**

***Index Terms*— l_p metric, Lee metric, perfect code, quasi-perfect code, tiling.**

I. Introduction

The Lee metric was introduced for the first time in [16] and [21] when dealing with transmission of signals over noisy channels. Since then several types of codes under the Lee metric were studied (see [1]–[4], [6]–[8], and [12]). In this paper, we focus on perfect and quasi-perfect l_p codes in $\mathbb{Z}^n$. Golomb and Welch [8] conjectured that the perfect codes under the Lee metric only exist for spheres of radius $r = 1$ or in Lee spaces of dimension $n = 1, 2$. Besides practical applications, the Golomb-Welch conjecture has been the main motive behind the research in the area for the last 45 years. Although there are many papers on the topic, the conjecture is far from being solved.

Gravier *et al.* [9] settled the Golomb-Welch conjecture for 3-dimensional Lee space. Dimension 4 was studied by Špacapan [19] with the aid of computer. It was proved in [10] that there are no perfect Lee codes for $3 \leq n \leq 5$ and $r > 1$. Horak [11] showed the nonexistence of perfect Lee codes for $n = 6$ and $r = 2$. Horak and Grošek [14] gave a new approach for tackling the conjecture and proved the nonexistence of linear perfect Lee codes for $7 \leq n \leq 11$ and $r = 2$.

Other researchers have considered the conjecture for large dimensions. Golomb and Welch [8] proved the nonexistence of perfect Lee codes for $r \geq r_n$, where r_n has not been specified. Later, Post [18] showed that there is no linear perfect code for $r \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2} - 2)$ and $n \geq 6$. Lepistö [17] proved that a perfect Lee code must satisfy $n \geq (r + 2)^2/2.1$, where $r \geq 285$.

Although the Golomb-Welch conjecture has not been solved yet, it is widely believed that it is true. Therefore, instead of searching for perfect Lee codes, some codes that are close to being perfect are considered [1]. Horak and Grošek [14] constructed some quasi-perfect Lee codes for $n = 3$. They also showed that there are at most finitely many values of r for which there exists a quasi-perfect r error correcting Lee code in $\mathbb{Z}^n$.

Recently, the notion of perfect Lee code has been extended to perfect l_p code, $p \geq 2$ [5]. It was shown that for $n = 2, 3$ and $p = 2$, there are linear perfect codes only for the parameters $n = 2$ with $r = 1, \sqrt{2}, 2, 2\sqrt{2}$ and $n = 3$ with $r = 1, \sqrt{3}$. It was also shown that for $n = 2$ and r being integer there are no perfect codes under the l_p metric if $r > 2$ and $2 \leq p < \infty$. Later, Strapasson *et al.* [20] considered the quasi-perfect codes and determined all radii for which there are linear quasi-perfect codes for $p = 2$ and $n = 2, 3$.

In this paper, we prove some nonexistence results of linear perfect l_p codes for $p = 1$ and $2 \leq p < \infty$, $r = 2^{1/p}, 3^{1/p}$. We also give an algebraic construction of quasi-perfect l_p codes for $p = 1, r = 2$ and $2 \leq p < \infty, r = 2^{1/p}$. This paper is organized as follows. In Section II, we give some definitions and basic results about codes in $\mathbb{Z}^n$ under the l_p metric. In Section III, we show the nonexistence of certain perfect l_p codes. In Section IV, we present some constructions of quasi-perfect l_p codes. Section V concludes the paper.

II. Preliminaries

A code C in $\mathbb{Z}^n$ is a subset of $\mathbb{Z}^n$. If a code C is at the same time a lattice then C is called a linear code. Linear codes play a special role as in this case there is a better chance for the existence of an efficient decoding algorithm. The l_p distance between two points $x = (x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n, = (y_1, y_2, \cdots, y_n) \in \mathbb{Z}^n$ is defined by

$$d_p(x, y) := (\sum_{i=1}^{n} |x_i - y_i|^p)^{\frac{1}{p}},$$

where $1 \leq p < \infty$. If $p = 1$, then the l_1 distance is also called Lee distance, and the corresponding code is called Lee code. The minimum distance $d_p(C)$ of a code C in $\mathbb{Z}^n$ is defined by

$$d_p(C) := \min\{d_p(x, y) : \ x, y \in C, \ x \neq y\}.$$

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The ball in $\mathbb{Z}^n$ centered at $x = (x_1, x_2, \cdots, x_n)$ with radius r is defined by

$$B_p^n(x, r) := \{z \in \mathbb{Z}^n : \ d_p(x, z) \le r\}.$$

When $x = 0$ we will denote $B_p^n(0, r)$ by $B_p^n(r)$.

In order to define the packing and the covering radius of $C \subseteq \mathbb{Z}^n$ under the l_p metric for $1 \le p < \infty$, we first define the distance set of the l_p metric in $\mathbb{Z}^n$ as

$$D_{p,n} = \{d \in \mathbb{R} : \text{ there are } z_1, z_2 \in \mathbb{Z}^n \text{ with } d_p(z_1, z_2) = d\}.$$

It is easy to see that $D_{p,n} \subseteq \{0, 1, 2^{1/p}, 3^{1/p}, \cdots\}$ and $D_{1,n} = \{0, 1, 2, 3, \cdots\}$. $D_{2,n}$ has been determined in [5], but in general, it is not an easy task to determine the set $D_{p,n}$. In the following, we denote the elements of the set $D_{p,n}$ by

$$D_{p,n} = \{d_{p,n,i} : \ i = 0, 1, 2, \cdots, \text{ and } d_{p,n,0} < d_{p,n,1} < d_{p,n,2} < \cdots\}.$$

The packing radius of a code $C \subset \mathbb{Z}^n$ under the l_p metric is the greatest $r \in D_{p,n}$ such that $B_p^n(x, r) \bigcap B_p^n(y, r) = \emptyset$ holds for all $x, y \in C$. The packing radius of a code $C \subset \mathbb{Z}^n$ under the l_p metric will be denoted by $r_p(C)$.

The covering radius of a code $C \subset \mathbb{Z}^n$ under the l_p metric is the smallest $r \in D_{p,n}$ such that $\bigcup_{c \in C}(c + B_p^n(r)) = \mathbb{Z}^n$. The covering radius of a code $C \subset \mathbb{Z}^n$ under the l_p metric will be denoted by $R_p(C)$.

An (n, r) code $C \subseteq \mathbb{Z}^n$ is called perfect under l_p metric if $r_p(C) = R_p(C) = r$. An (n, r) code $C \subseteq \mathbb{Z}^n$ is called quasi-perfect under l_p metric if there exists an integer i such that $r_p(C) = r = d_{p,n,i}$ and $R_p(C) = d_{p,n,i+1}$.

Another way of introducing a perfect l_p code is by means of a tiling. Let V be a subset of $\mathbb{Z}^n$. By a copy of V we mean a translation $V + x = \{v + x : \ v \in V\}$ of V, where $x \in \mathbb{Z}^n$. A collection $\mathfrak{T} = \{V + l : \ l \in L\}$, $L \subseteq \mathbb{Z}^n$, of copies of V constitutes a tiling of $\mathbb{Z}^n$ by V if $\mathfrak{T}$ forms a partition of $\mathbb{Z}^n$. $\mathfrak{T}$ is called a lattice tiling if L forms a lattice. Clearly, a code C is perfect under l_p metric if and only if $\{B_p^n(r_p(C)) + c : \ c \in C\}$ constitutes a tiling of $\mathbb{Z}^n$ by $B_p^n(r_p(C))$.

Note that if there exist $V \subseteq \mathbb{Z}^n$ and $C \subseteq \mathbb{Z}^n$ such that $B_p^n(d_{p,n,i}) \subsetneqq V \subsetneqq B_p^n(d_{p,n,i+1})$ for some i and $\{V + c : \ c \in C\}$ constitutes a tiling of $\mathbb{Z}^n$ by V, then the set C is a quasi-perfect l_p code in $\mathbb{Z}^n$, and we will denote such a code by quasi-perfect $(n, d_{p,n,i}, |V|)$ code under l_p metric.

The following theorem can be found in [13].

Theorem 1 [13]: Let $S \subseteq \mathbb{Z}^n$ such that $|S| = m$. There is a lattice tiling of $\mathbb{Z}^n$ by translates of S if and only if there are both an Abelian group G of order m and a homomorphism $\phi : \mathbb{Z}^n \mapsto G$ such that the restriction of ϕ to S is a bijection.

The following theorem is the l_p metric version of [14, Th. 11], and it can be seen as a corollary of Theorem 1.

Theorem 2: There is a linear quasi-perfect $(n, d_{p,n,i}, M)$ code under l_p metric for any integer i if there are both an Abelian group G of order M with $|B_p^n(d_{p,n,i})| < M < |B_p^n(d_{p,n,i+1})|$ and a homomorphism $\phi : \mathbb{Z}^n \mapsto G$ such that the restriction of ϕ to $B_p^n(d_{p,n,i})$ is an injection and the restriction of ϕ to $B_p^n(d_{p,n,i+1})$ is a surjection.

III. Nonexistence Results

A. $p = 1$

In this subsection, we study the nonexistence of perfect Lee codes. The size of $B_1^n(r)$ is well known [8] and we denote it by $k_{n,r}$:

$$k_{n,r} := |B_1^n(r)| = \sum_{i=0}^{\min\{n,r\}} 2^i \binom{n}{i}\binom{r}{i}.$$

In order to get our main result, we need the following lemmas.

Lemma 3: $\sum_{c_1,\cdots,c_n \in\{\pm 1\}} (\sum_{i=1}^n c_i b_i)^2 = 2^n \sum_{i=1}^n b_i^2$.

Proof:

$$\begin{aligned}
&\sum_{c_1,\cdots,c_n\in\{\pm1\}} (\sum_{i=1}^n c_i b_i)^2 \\
&= \sum_{c_1,\cdots,c_n\in\{\pm1\}} (\sum_{i=1}^n b_i^2 + 2\sum_{1\le i<j\le n} c_i b_i c_j b_j) \\
&= 2^n \sum_{i=1}^n b_i^2 + 2 \sum_{c_1,\cdots,c_n\in\{\pm1\}} \sum_{1\le i<j\le n} c_i c_j b_i b_j \\
&= 2^n \sum_{i=1}^n b_i^2 + 2 \sum_{1\le i<j\le n} (1\cdot 1\cdot b_i b_j + 1\cdot(-1)\cdot b_i b_j + \\
&\quad (-1)\cdot 1\cdot b_i b_j + (-1)\cdot(-1)\cdot b_i b_j) \\
&= 2^n \sum_{i=1}^n b_i^2.
\end{aligned}$$

■

Lemma 4 ([22, Th. 13.1 and Problem 13E]): The number of solutions of $x_1 + x_2 + \cdots + x_k \le n$ in $\mathbb{Z}_{>0}$ is $\binom{n}{k}$.

Lemma 5:

$$\sum_{i=1}^t \sum_{\substack{x_i \ge 1 \\ \sum_{i=1}^t x_i \le r}} x_i^2 y_i = \sum_{j=1}^{r-t+1} j^2 \binom{r-j}{t-1} \sum_{i=1}^t y_i.$$

Proof: Since $x_i \ge 1$ and $\sum_{i=1}^t x_i \le r$, then $1 \le x_i \le r - t + 1$ for $1 \le i \le t$. For each $1 \le j \le r - t + 1$, by Lemma 4, the number of solution of $x_i = j$ and $\sum_{\substack{1\le k\le t \\ k\ne i}} x_k \le r - j$ in $\mathbb{Z}_{>0}$ is $\binom{r-j}{t-1}$. Hence $\sum_{i=1}^t \sum_{\substack{x_i \ge 1 \\ \sum_{i=1}^t x_i \le r}} x_i^2 y_i = \sum_{j=1}^{r-t+1} j^2 \binom{r-j}{t-1} \sum_{i=1}^t y_i$. ■

Lemma 6: $\sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{i=1}^t x_{l_i} = \binom{n-1}{t-1} \sum_{i=1}^n x_i$.

Proof: Since $1 \le l_1 < l_2 < \cdots < l_t \le n$, if $i \in \{l_1, l_2, \cdots, l_t\}$, then there are $\binom{n-1}{t-1}$ choices for other $t-1$ numbers. Hence the identity follows from the fact that for $1 \le i \le n$, x_i occurs on the left hand side $\binom{n-1}{t-1}$ times. ■

Now we state our main result.

Theorem 7: Let $r \le n$, $p_{n,r} = \sum_{t=1}^r 2^t \sum_{j=1}^{r-t+1} j^2 \binom{r-j}{t-1}\binom{n-1}{t-1}$. If $k_{n,r} \equiv 3$ or $6 \pmod 9$, $p_{n,r} \equiv 0 \pmod 3$ and $k_{n,r}$ is squarefree, then there does not exist a linear perfect (n, r) Lee code.

Proof: If $k_{n,r} = |B_1^n(r)| \equiv 3$ or $6 \pmod 9$, $p_{n,r} \equiv 0 \pmod 3$ and $k_{n,r}$ is squarefree. Then each Abelian group of order $k_{n,r}$ is isomorphic to the cyclic group $\mathbb{Z}_{k_{n,r}}$. From Theorem 1, we need to show that there is no homomorphism $\phi : \mathbb{Z}^n \mapsto \mathbb{Z}_{k_{n,r}}$ such that the restriction of ϕ to

$B_1^n(r)$ is a bijection. Note that each homomorphism $\phi : \mathbb{Z}^n \mapsto \mathbb{Z}_{k_{n,r}}$ is determined by the values of $\phi(e_i)$, $i = 1, \cdots, n$, where e_i, $i = 1, \cdots, n$, is the standard basis of $\mathbb{Z}^n$. If $\{\sum_{i=1}^t \pm b_{l_i}\phi(e_{l_i}) : 1 \le t \le r, 1 \le l_1 < l_2 < \cdots < l_t \le n, b_{l_i} \ge 1, \sum_{i=1}^t b_{l_i} \le r\} \ne \mathbb{Z}_{k_{n,r}} \backslash \{0\}$, then ϕ is not a bijection on $B_1^n(r)$. Hence, it is sufficient to show that for each n-tuple $(a_1, \cdots, a_n)$ of elements in $\mathbb{Z}_{k_{n,r}}$,

$$\Big\{\sum_{i=1}^t \pm b_{l_i}a_{l_i} : 1 \le t \le r, 1 \le l_1 < l_2 < \cdots < l_t \le n, b_{l_i} \ge 1, \sum_{i=1}^t b_{l_i} \le r\Big\} \ne \mathbb{Z}_{k_{n,r}} \backslash \{0\}.$$

If not, we have

$$\sum_{t=1}^r \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{\substack{b_{l_i}\ge 1\\ \sum_{i=1}^t b_{l_i}\le r}} (\sum_{i=1}^t \pm b_{l_i}a_{l_i})^2 \equiv \sum_{i=1}^{k_{n,r}-1} i^2 \pmod{k_{n,r}}. \tag{1}$$

Then the first formula of Eq. (1) can be written as

$$\begin{aligned}
&\sum_{t=1}^r \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{\substack{b_{l_i}\ge 1\\ \sum_{i=1}^t b_{l_i}\le r}} \sum_{c_{l_1},\cdots,c_{l_t}\in\{\pm1\}} (\sum_{i=1}^t c_{l_i}b_{l_i}a_{l_i})^2\\
&= \sum_{t=1}^r 2^t \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{\substack{b_{l_i}\ge 1\\ \sum_{i=1}^t b_{l_i}\le r}} \sum_{i=1}^t b_{l_i}^2 a_{l_i}^2\\
&= \sum_{t=1}^r 2^t \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{i=1}^t \sum_{\substack{b_{l_i}\ge 1\\ \sum_{i=1}^t b_{l_i}\le r}} b_{l_i}^2 a_{l_i}^2\\
&= \sum_{t=1}^r 2^t \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{j=1}^{r-t+1} j^2\binom{r-j}{t-1} \sum_{i=1}^t a_{l_i}^2\\
&= \sum_{t=1}^r 2^t \sum_{j=1}^{r-t+1} j^2\binom{r-j}{t-1} \sum_{1\le l_1<l_2<\cdots<l_t\le n} \sum_{i=1}^t a_{l_i}^2\\
&= \sum_{t=1}^r 2^t \sum_{j=1}^{r-t+1} j^2\binom{r-j}{t-1}\binom{n-1}{t-1}(\sum_{i=1}^n a_i^2)\\
&= p_{n,r}(\sum_{i=1}^n a_i^2),
\end{aligned}$$

where the first equation follows from Lemma 3, the third from Lemma 5, and the fifth from Lemma 6. By the well known expression for the sum of squares, we get that $\sum_{i=1}^{k_{n,r}-1} i^2 = \frac{(k_{n,r}-1)k_{n,r}(2k_{n,r}-1)}{6}$. By hypothesis, we have $3 \mid k_{n,r}$, $3 \mid p_{n,r}$, and it is easy to see that $3 \nmid \frac{(k_{n,r}-1)k_{n,r}(2k_{n,r}-1)}{6}$ if $k_{n,r} \equiv 3$ or $6 \pmod 9$, which contradicts Eq. (1), that is $p_{n,r}(\sum_{i=1}^n a_i^2) \equiv \frac{(k_{n,r}-1)k_{n,r}(2k_{n,r}-1)}{6} \pmod{k_{n,r}}$. ■

By considering the radii $r = 3$ and $r = 4$ in the last theorem we get the following corollaries.

Corollary 8: If $n \equiv 12$ *or* $21 \pmod{27}$ *and* $k_{n,3}$ *is squarefree, then there does not exist a linear perfect* $(n, 3)$ *Lee code.*

Proof: By Theorem 7, we need to prove that $k_{n,3} \equiv 3$ or $6 \pmod 9$ and $p_{n,3} \equiv 0 \pmod 3$. By definition, we have

$$\begin{aligned}
k_{n,3} &= \sum_{i=0}^3 2^i\binom{n}{i}\binom{3}{i}\\
&= 1 + 6n + 4\binom{n}{2}\binom{3}{2} + 8\binom{n}{3}\\
&= 1 + 6n^2 + \frac{4n(n-1)(n-2)}{3}.
\end{aligned}$$

Then $k_{n,3} \equiv 3$ or $6 \pmod 9$ is equivalent to $3 + 18n^2 + 4n(n-1)(n-2) \equiv 9$ or $18 \pmod{27}$. Hence $k_{n,3} \equiv 3$ or $6 \pmod 9$ if and only if $n \equiv 1, 11, 12, 19, 20$ or $21 \pmod{27}$.

Similarly, we have

$$\begin{aligned}
p_{n,3} &= \sum_{t=1}^3 2^t \sum_{j=1}^{4-t} j^2\binom{3-j}{t-1}\binom{n-1}{t-1}\\
&= 2\sum_{j=1}^3 j^2 + 4\sum_{j=1}^2 j^2\binom{3-j}{1}\binom{n-1}{1} + 8\binom{n-1}{2}\\
&= 28 + 24(n-1) + 4(n-1)(n-2).
\end{aligned}$$

Then $p_{n,3} \equiv 0 \pmod 3$ if and only if $n \equiv 0 \pmod 3$.

Therefore only when $n \equiv 12$ or $21 \pmod{27}$, we have both $k_{n,3} \equiv 3$ or $6 \pmod 9$ and $p_{n,3} \equiv 0 \pmod 3$. ■

Corollary 9: If $n \equiv 3, 5, 21$ *or* $23 \pmod{27}$*,* $n \ge 4$ *and* $k_{n,4}$ *is squarefree, then there does not exist a linear perfect* $(n, 4)$ *Lee code.*

Proof: By Theorem 7, we need to prove that $k_{n,4} \equiv 3$ or $6 \pmod 9$ and $p_{n,4} \equiv 0 \pmod 3$. By definition, we have

$$\begin{aligned}
k_{n,4} &= \sum_{i=0}^4 2^i\binom{n}{i}\binom{4}{i}\\
&= 1 + 8n + 4\binom{n}{2}\binom{4}{2} + 8\binom{n}{3}\binom{4}{3} + 16\binom{n}{4}\\
&= 1 + 8n + 12n(n-1) + \frac{16n(n-1)(n-2)}{3}\\
&\quad + \frac{2n(n-1)(n-2)(n-3)}{3}.
\end{aligned}$$

Then $k_{n,4} \equiv 3$ or $6 \pmod 9$ is equivalent to $3 + 24n + 36n(n-1) + 16n(n-1)(n-2) + 2n(n-1)(n-2)(n-3) \equiv 9$ or $18 \pmod{27}$. Hence $k_{n,4} \equiv 3$ or $6 \pmod 9$ if and only if $n \equiv 3, 4, 5, 13, 21, 22$ or $23 \pmod{27}$.

Similarly, we can compute $p_{n,4}$ as Eq. (2) on the top of next page.

Then $p_{n,4} \equiv 0 \pmod 3$ is equivalent to $180 + 240(n-1) + 84(n-1)(n-2) + 8(n-1)(n-2)(n-3) \equiv 0 \pmod 9$. Hence $p_{n,4} \equiv 0 \pmod 3$ if and only if $n \equiv 1, 3$ or $5 \pmod 9$.

Therefore only when $n \equiv 3, 5, 21$ or $23 \pmod{27}$, we have both $k_{n,4} \equiv 3$ or $6 \pmod 9$ and $p_{n,4} \equiv 0 \pmod 3$. ■

Remark 10: When $r = 2$*, then* $k_{n,2} = 2n^2 + 2n + 1$ *and* $p_{n,2} = 4n + 6$*, there is no integer* n *satisfying the conditions of Theorem 7. Hence we can not get any nonexistence result for the linear perfect* $(n, 2)$ *Lee code from Theorem 7.*

$$
\begin{aligned}
p_{n,4} &= \sum_{t=1}^{4} 2^t \sum_{j=1}^{5-t} j^2 \binom{4-j}{t-1}\binom{n-1}{t-1} \\
&= 2\sum_{j=1}^{4} j^2 + 4\sum_{j=1}^{3} j^2 \binom{4-j}{1}\binom{n-1}{1} + 8\sum_{j=1}^{2} j^2 \binom{4-j}{2}\binom{n-1}{2} + 16\binom{n-1}{3} \\
&= \frac{8(n-1)(n-2)(n-3)}{3} + 4(n-1)(n-2)\sum_{j=1}^{2} j^2 \binom{4-j}{2} + 4(n-1)\sum_{j=1}^{3} j^2(4-j) + 60 \\
&= 60 + 80(n-1) + 28(n-1)(n-2) + \frac{8(n-1)(n-2)(n-3)}{3} \qquad (2)
\end{aligned}
$$

TABLE I
NONEXISTENCE OF LINEAR PERFECT (n, r) LEE CODES

r	n
3	21, 39, 48, 66, 75, 93, 120, 129, 156, 174, 183, 201, 210, 228, 255, 291
4	5, 21, 23, 32, 48, 50, 59, 75, 77, 84, 86, 102, 104, 111, 113, 129, 131, 138

Table I gives the first integers that satisfy the conditions of Theorem 7. In fact, we find that there are 265 integers satisfying the conditions of Theorem 7 when $r = 3$ and $n \le 5000$, and 734 integers satisfying the conditions of Theorem 7 when $r = 4$ and $n \le 5000$. It seems that there are many parameters (n, r) satisfying the conditions in Theorem 7.

Remark 11: Horak and Grošek [14] conjectured that, for each $n \ge 2$ and $r > 0$ if there are both an Abelian group G of order $|B_1^n(r)|$ and a homomorphism $\phi : \mathbb{Z}^n \mapsto G$ such that the restriction of ϕ to $B_1^n(r)$ is a bijection, then there is a homomorphism $\phi : \mathbb{Z}^n \mapsto \mathbb{Z}_{|B_1^n(r)|}$ such that the restriction of ϕ to $B_1^n(r)$ is a bijection. If the conjecture is right, then we do not need the condition $k_{n,r}$ is squarefree in Theorem 7.

B. $2 \le p < \infty$

In contrast to Lee metric, there exist perfect l_p $(2 \le p < \infty)$ codes for infinitely many radii and dimensions. For example, there are perfect $(n, n^{\frac{1}{p}} r)$ under the l_p metric for $n < (1 + 1/r)^p$ [5]. In this subsection, we study the nonexistence of perfect l_p $(2 \le p < \infty)$ codes with small radii. Note that little is known on the number of points of the balls $B_p^n(r)$ for general n and r, but the radii $2^{\frac{1}{p}}$ and $3^{\frac{1}{p}}$ are very special. Since when the dimension n is fixed, the balls in the l_p metric with these radii are the same for any p, and it is easy to see that

$$
\begin{aligned}
k_{n,2,p} &:= |B_p^n(2^{\frac{1}{p}})| = 2n^2 + 1, \text{ and} \\
k_{n,3,p} &:= |B_p^n(3^{\frac{1}{p}})| = 1 + 2n^2 + \frac{4n(n-1)(n-2)}{3}.
\end{aligned}
$$

Theorem 12: If $n \equiv 5$ or $8 \pmod 9$ and $k_{n,2,p}$ is squarefree, then there does not exist a linear perfect $(n, 2^{1/p})$ code under l_p metric.

Proof: The proof is similar to that of Theorem 7, we only need to show that for each n-tuple $(a_1, \cdots, a_n)$ of elements in $\mathbb{Z}_{k_{n,2,p}}$,

$$
\left\{ \pm a_i, \pm a_j \pm a_k : \ 1 \le i \le n, \ 1 \le j < k \le n \right\} \ne \mathbb{Z}_{k_{n,2,p}} \backslash \{0\}.
$$

If not, we have

$$
\begin{aligned}
2\sum_{i=1}^{n} a_i^2 + 2\sum_{1 \le i < j \le n} ((a_i + a_j)^2 + (a_i - a_j)^2) \\
\equiv \sum_{i=1}^{k_{n,2,p}-1} i^2 \pmod{k_{n,2,p}}.
\end{aligned}
$$

That is

$$
(4n-2)\sum_{i=1}^{n} a_i^2 \equiv \frac{(k_{n,2,p}-1)k_{n,2,p}(2k_{n,2,p}-1)}{6} \pmod{k_{n,2,p}}.
$$

If $n \equiv 5$ or $8 \pmod 9$, then $k_{n,2,p} \equiv 3$ or $6 \pmod 9$ and $3 \mid (4n-2)$. Hence $3 \mid k_{n,2,p}$ and $3 \nmid \frac{(k_{n,2,p}-1)k_{n,2,p}(2k_{n,2,p}-1)}{6}$, which is a contradiction. ■

Theorem 13: If $n \equiv 11, 12, 20$ or $21 \pmod{27}$ and $k_{n,3,p}$ is squarefree, then there does not exist a linear perfect $(n, 3^{1/p})$ code under l_p metric.

Proof: The proof is similar to that of Theorem 7, we only need to show that for each n-tuple $(a_1, \cdots, a_n)$ of elements in $\mathbb{Z}_{k_{n,3,p}}$,

$$
\begin{aligned}
&\big\{ \pm a_i, \pm a_{j_1} \pm a_{j_2}, \pm a_{l_1} \pm a_{l_2} \pm a_{l_3} : \ 1 \le i \le n, \\
&\quad 1 \le j_1 < j_2 \le n, \ 1 \le l_1 < l_2 < l_3 \le n \big\} \ne \mathbb{Z}_{k_{n,3,p}} \backslash \{0\}.
\end{aligned}
$$

If not, we have

$$
\begin{aligned}
2\sum_{i=1}^{n} a_i^2 &+ 2\sum_{1 \le i < j \le n} ((a_i + a_j)^2 + (a_i - a_j)^2) \\
&+ 2\sum_{1 \le i < j < k \le n} ((a_i + a_j + a_k)^2 + (a_i + a_j - a_k)^2 \\
&+ (a_i - a_j + a_k)^2 + (a_i - a_j - a_k)^2) \\
&\equiv \sum_{i=1}^{k_{n,3,p}-1} i^2 \pmod{k_{n,3,p}}.
\end{aligned}
$$

That is

$$(4n^2-8n+6)\sum_{i=1}^{n} a_i^2 \equiv \frac{(k_{n,3,p}-1)k_{n,3,p}(2k_{n,3,p}-1)}{6} \pmod{k_{n,3,p}}.$$

If $n \equiv 11, 12, 20$ or $21 \pmod{27}$, then $k_{n,3,p} \equiv 3$ or $6 \pmod 9$ and $3 \mid (4n^2-8n+6)$. Hence $3 \mid k_{n,3,p}$ and $3 \nmid \frac{(k_{n,3,p}-1)k_{n,3,p}(2k_{n,3,p}-1)}{6}$, which is a contradiction. ■

Table II gives some parameters of the nonexistence of linear perfect l_p codes, where $2 \le p < \infty$. In fact, we find that there are 1073 integers satisfying the conditions of Theorem 12 when $r = 2^{1/p}$ and $n \le 5000$, and 701 integers satisfying the conditions of Theorem 13 when $r = 3^{1/p}$ and $n \le 5000$.

Remark 14: It should be noted that we can get similar results for other small radii, but it is difficult to get the general result since we know little about the set $D_{p,n}$ and the structure of $B_p^n(r)$, where $2 \le p < \infty$.

Remark 15:
1) *If we denote the multiplicative semi-group of the ring $\mathbb{Z}_m$ as R_m, then the main technique used in the above two subsections is to choose a homomorphism $\chi : R_m \to R_m$, where $\chi(a) = a^2$. More results of the same type may be obtained by using other homomorphisms.*
2) *As we have pointed out that our method does not work for linear perfect $(n, 2)$ Lee codes in Remark 10. Recently, Kim [15] proved some nonexistence results for linear perfect $(n, 2)$ Lee codes by choosing $\chi(a) = a^{2k}$.*

IV. Quasi-Perfect l_p Codes

In this section, we give an algebraic construction of quasi-perfect $(n, 2, q)$ Lee codes and quasi-perfect $(n, 2^{1/p}, q)$ codes under l_p metric. From Theorem 2, there exists a linear quasi-perfect $(n, 2, q)$ Lee code, if $|B_1^n(2)| < q < |B_1^n(3)|$ and there are both an Abelian group G of order q and a homomorphism $\phi : \mathbb{Z}^n \mapsto G$ such that the restriction of ϕ to $B_1^n(2)$ is an injection and the restriction of ϕ to $B_1^n(3)$ is a surjection. Similarly, there exists a quasi-perfect $(n, 2^{1/p}, q)$ code under l_p metric, if $|B_p^n(2^{\frac{1}{p}})| < q < |B_p^n(3^{\frac{1}{p}})|$ and there are both an Abelian group G of order q and a homomorphism $\phi : \mathbb{Z}^n \mapsto G$ such that the restriction of ϕ to $B_p^n(2^{\frac{1}{p}})$ is an injection and the restriction of ϕ to $B_p^n(3^{\frac{1}{p}})$ is a surjection. We also mention that for the quasi-perfect $(n, 2, q)$ Lee code, if the dimension n is fixed, then the smaller q is, the closer the code is to be perfect. Similar for the quasi-perfect $(n, 2^{1/p}, q)$ code $C \subseteq \mathbb{Z}^n$ under l_p metric.

A. p=1

Theorem 16: Let $q = 2nm + 1$ be a prime number and n, m be integers with $n \equiv 1 \pmod 6$, $n \ge 7$ and $n + 1 < m < 3n + \frac{2(n-1)(n-2)}{3}$. Let g be a primitive root modulo q, we denote

$$S := \{1,2\} \bigcup \{1+g^{2mk}, 1-g^{2mk} : 1 \le k \le \frac{n-1}{2}\},$$

$$T := \{1,2,3\} \bigcup \{1+g^{2mk}, 1-g^{2mk}, 1+2g^{2mk}, 1-2g^{2mk}, 2+g^{2mk}, 2-g^{2mk} : 1 \le k \le \frac{n-1}{2}\} \bigcup \{1+g^{2mk}+g^{2ml}, 1+g^{2mk}-g^{2ml}, 1-g^{2mk}+g^{2ml}, 1-g^{2mk}-g^{2ml} : 1 \le k \le \frac{n-1}{3},\ 2k \le l \le n-1-k\}.$$

If $|\{ind_g(i) \pmod m : i \in S\}| = n + 1$ and $|\{ind_g(i) \pmod m : i \in T\backslash\{0\}\}| = m$, then there exists a quasi-perfect $(n, 2, q)$ Lee code.

Proof: It is easy to see that $|B_1^n(2)| < q < |B_1^n(3)|$. In the following, we show that there is a homomorphism $\phi : \mathbb{Z}^n \mapsto \mathbb{Z}_q$ such that the restriction of ϕ to $B_1^n(2)$ is an injection and the restriction of ϕ to $B_1^n(3)$ is a surjection. Then it is sufficient to show that there exists an n-tuple $(a_1, \cdots, a_n)$ of elements in $\mathbb{Z}_q$ such that

$$|\{0, \pm a_i, \pm 2a_i, \pm a_j \pm a_k : 1 \le i \le n,\ 1 \le j < k \le n\}| = k_{n,2}, \text{ (injection)}$$

$$\{0, \pm a_i, \pm 2a_i, \pm 3a_i, \pm a_{j_1} \pm a_{j_2}, \pm 2a_{j_1} \pm a_{j_2}, \pm a_{j_1} \pm 2a_{j_2}, \pm a_{k_1} \pm a_{k_2} \pm a_{k_3} : 1 \le i \le n, 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} = \mathbb{Z}_q \text{ (surjection)}.$$

Let $a_i = g^{2mi}$, $i = 1, 2, \cdots, n$. Then we have

$$\{\pm g^{2mi}, \pm 2g^{2mi}, \pm g^{2mj} \pm g^{2mk} : 1 \le i \le n,\ 1 \le j < k \le n\} = S \cdot \{\pm g^{2mi} : 1 \le i \le n\} = S \cdot \{g^{mi} : 1 \le i \le 2n\}.$$

Since $|S| = n + 1$ and $|\{\text{ind}_g(i) \pmod m : i \in S\}| = n + 1$, then $|S \cdot \{g^{mi} : 1 \le i \le 2n\}| = 2n(n+1) = k_{n,2} - 1$. Note that $0 \notin S \cdot \{g^{mi} : 1 \le i \le 2n\}$, hence $|\{0, \pm g^{2mi}, \pm 2g^{2mi}, \pm g^{2mj} \pm g^{2mk} : 1 \le i \le n,\ 1 \le j < k \le n\}| = k_{n,2}$. We can also obtain

$$\{\pm g^{2mi}, \pm 2g^{2mi}, \pm 3g^{2mi}, \pm g^{2mj_1} \pm g^{2mj_2}, \pm 2g^{2mj_1} \pm g^{2mj_2}, \pm g^{2mj_1} \pm 2g^{2mj_2}, \pm g^{2mk_1} \pm g^{2mk_2} \pm g^{2mk_3} : 1 \le i \le n, 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} \supseteq (T\backslash\{0\}) \cdot \{\pm g^{2mi} : 1 \le i \le n\} \supseteq (T\backslash\{0\}) \cdot \{g^{mi} : 1 \le i \le 2n\} \supseteq \mathbb{Z}_q \backslash \{0\},$$

since $|\{\text{ind}_g(i) \pmod m : i \in T\backslash\{0\}\}| = m$. Hence

$$\{0, \pm g^{2mi}, \pm 2g^{2mi}, \pm 3g^{2mi}, \pm g^{2mj_1} \pm g^{2mj_2}, \pm 2g^{2mj_1} \pm g^{2mj_2}, \pm g^{2mj_1} \pm 2g^{2mj_2}, \pm g^{2mk_1} \pm g^{2mk_2} \pm g^{2mk_3} : 1 \le i \le n,\ 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} = \mathbb{Z}_q.$$

■

Remark 17: Camarero and Martísnez [4] constructed quasi-perfect $(2[\frac{p}{4}], 2, p^2)$ Lee codes for any prime $p \ge 7$ with $p \equiv \pm 5 \pmod{12}$. Theorem 16 gives a class of quasi-perfect Lee codes with new parameters since the dimension of the quasi-perfect Lee code in Theorem 16 is odd.

TABLE II

NONEXISTENCE OF LINEAR PERFECT (n, r) CODES UNDER l_p METRIC, WHERE $2 \le p < \infty$

r	n
$2^{1/p}$	5, 8, 14, 17, 23, 26, 32, 35, 41, 44, 50, 53, 59, 62, 68, 71, 77, 80, 86, 89, 95, 98
$3^{1/p}$	11, 12, 20, 21, 38, 39, 47, 48, 65, 66, 74, 75, 92, 93

TABLE III

QUASI-PERFECT $(n, 2, q)$ LEE CODES

n	7	7	19	19	25	25	25	25	31	31	31
q	197	211	2129	2357	5651	5701	5851	6451	4093	5333	7937
g	2	2	3	2	2	2	2	3	2	2	3

Table III lists some examples of quasi-perfect $(n, 2, q)$ Lee codes. Let us consider the quality of some of the constructed quasi-perfect Lee codes.

Example 18: Let $n = 7,\ q = 197,\ g = 2$, by Theorem 16, we have a quasi-perfect $(7, 2, 197)$ Lee code C with $|V| = 197$. Note that $|B_1^7(2)| = 113$ and $|B_1^7(3)| = 575$, then $|V| = 197$ is nearer to the number of points of the packing ball than to covering ball, hence the code is very close to be perfect.

Example 19: Let $n = 31,\ q = 4093,\ g = 2$, by Theorem 16, we have a quasi-perfect $(31, 2, 4093)$ Lee code C with $|V| = 4093$. Note that $|B_1^{31}(2)| = 1985$ and $|B_1^{31}(3)| = 41727$, then $|V| = 4093$ is nearer to the number of points of the packing ball than to covering ball, hence the code is very close to be perfect.

B. $2 \le p < \infty$

Theorem 20: Let $q = 2nm + 1$ be a prime number and n, m be integers with $n \equiv 1 \pmod 6$, $n \ge 7$ and $n + 1 \le m < n + \frac{2(n-1)(n-2)}{3}$. Let g be a primitive root modulo q, we denote

$$S := \{1\} \bigcup \{1 + g^{2mk}, 1 - g^{2mk} :\ 1 \le k \le \frac{n-1}{2}\},$$

$$\begin{aligned} T := \{1\} &\bigcup \{1 + g^{2mk}, 1 - g^{2mk} :\ 1 \le k \le \frac{n-1}{2}\} \\ &\bigcup \{1 + g^{2mk} + g^{2ml}, 1 + g^{2mk} - g^{2ml}, 1 - g^{2mk} + g^{2ml}, \\ &1 - g^{2mk} - g^{2ml} : 1 \le k \le \frac{n-1}{3},\ 2k \le l \le n-1-k\}. \end{aligned}$$

If $|\{ind_g(i) \pmod m) : i \in S\}| = n$ and $|\{ind_g(i) \pmod m) : i \in T\backslash\{0\}\}| = m$, then there exists a quasi-perfect $(n, 2^{1/p}, q)$ code under l_p metric.

Proof: The proof is similar to that of Theorem 16, we only need to show that there exists an n-tuple $(a_1, \cdots, a_n)$ of elements in $\mathbb{Z}_q$ such that

$$|\{0, \pm a_i, \pm a_j \pm a_k, :\ 1 \le i \le n,\ 1 \le j < k \le n\}| = k_{n,2,p},$$

$$\{0, \pm a_i, \pm a_{j_1} \pm a_{j_2}, \pm a_{k_1} \pm a_{k_2} \pm a_{k_3} :\ 1 \le i \le n,\ 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} = \mathbb{Z}_q.$$

Let $a_i = g^{2mi}$, $i = 1, 2, \cdots, n$. Then we have

$$\begin{aligned} &\{\pm g^{2mi}, \pm g^{2mj} \pm g^{2mk} :\ 1 \le i \le n,\ 1 \le j < k \le n\} \\ &= S \cdot \{\pm g^{2mi} :\ 1 \le i \le n\} \\ &= S \cdot \{g^{mi} :\ 1 \le i \le 2n\}. \end{aligned}$$

Since $|S| = n$ and $|\{\text{ind}_g(i) \pmod m) : i \in S\}| = n$, then $|S \cdot \{g^{mi} :\ 1 \le i \le 2n\}| = 2n^2 = k_{n,2,p} - 1$. Note that $0 \notin S \cdot \{g^{mi} :\ 1 \le i \le 2n\}$, hence $|\{0, \pm g^{2mi}, \pm g^{2mj} \pm g^{2mk}, :\ 1 \le i \le n,\ 1 \le j < k \le n\}| = k_{n,2,p}$. We can also obtain

$$\begin{aligned} &\{\pm g^{2mi}, \pm g^{2mj_1} \pm g^{2mj_2}, \pm g^{2mk_1} \pm g^{2mk_2} \pm g^{2mk_3} : \\ &\quad 1 \le i \le n,\ 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} \\ &\quad \supseteq (T\backslash\{0\}) \cdot \{\pm g^{2mi} :\ 1 \le i \le n\} \\ &\quad \supseteq (T\backslash\{0\}) \cdot \{g^{mi} :\ 1 \le i \le 2n\} \\ &\quad \supseteq \mathbb{Z}_q\backslash\{0\}, \end{aligned}$$

since $|\{\text{ind}_g(i) \pmod m) : i \in T\backslash\{0\}\}| = m$. Hence

$$\{0, \pm g^{2mi}, \pm g^{2mj_1} \pm g^{2mj_2}, \pm g^{2mk_1} \pm g^{2mk_2} \pm g^{2mk_3} :\ 1 \le i \le n,\ 1 \le j_1 < j_2 \le n,\ 1 \le k_1 < k_2 < k_3 \le n\} = \mathbb{Z}_q.$$

■

TABLE IV

QUASI-PERFECT $(n, 2^{1/p}, q)$ CODES UNDER l_p METRIC, WHERE $2 \le p < \infty$

n	19	19	25	31	31	31	31	31
q	2129	2357	5651	4093	5333	6883	7937	8123
g	3	2	2	2	2	2	3	2

Table IV lists some examples of quasi-perfect $(n, 2^{1/p}, q)$ codes under l_p metric, where $2 \le p < \infty$. Now we give an example to consider the quality of the constructed quasi-perfect code under l_p metric.

Example 21: Let $n = 31$, $q = 4093$, $g = 2$, by Theorem 20, we have a quasi-perfect $(31, 2^{1/p}, 4093)$ code C under l_p metric with $|V| = 4093$. Note that $|B_p^{31}(2^{1/p})| = 1923$ and $|B_p^{31}(3^{\frac{1}{p}})| = 37883$, then $|V| = 4093$ is nearer to the number of points of the packing ball than to covering ball, hence the code is very close to be perfect.

V. CONCLUSION

In this paper, by studying the connections between perfect l_p codes and Abelian groups, we prove some nonexistence results of linear perfect l_p codes for $p = 1$ and $2 \le p < \infty$, $r = 2^{1/p}, 3^{1/p}$. In particular, we partially affirm the Golomb-Welch conjecture which states that there is no perfect r error correcting Lee code of length n for $n \geq 3$ and $r > 1$. Since it is widely believed that the Golomb-Welch conjecture is true, constructing codes close to perfect makes sense. In Section IV, we give an algebraic construction of quasi-perfect l_p codes for $p = 1, r = 2$ and $2 \leq p < \infty, r = 2^{1/p}$. It should be noted that similar methods may also work for other parameters. We also list some examples satisfying the conditions in Theorem 16 (and Theorem 20), leaving an open question whether there are infinitely many primes satisfying these conditions.

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